ON A RESULT OF G. D. BIRKHOFF ON LINEAR DIFFERENTIAL SYSTEMS

P. MASANI

We give a simple example to show that a result on the equivalent singular points of systems of ordinary linear differential equations due to G. D. Birkhoff [3; **5**, pp. 252–257] needs amendment. In matrix notation, in which Y, P, etc. denote $n\times n$ matrix-valued functions of a complex variable z, this result is as follows.

A. **Result.** *Every linear differential system*

$$Y'(z) = P(z)Y(z) \tag{1}$$

with a singular point of rank $q+1$ *at* $z=\infty$ $(q\geqq -1)$ *is equivalent at* $z=\infty$ *to a canonical system*

$$\overline{Y}'(z) = \overline{P}(z)\overline{Y}(z) \tag{$\bar{1}$}$$

in which $z\overline{P}(z)$ *is a polynomial of degree less than or equal to* $q+1$.

B. **Definitions.**[1] (a) *The equation* (1) *is said to have a* singular point of rank $q+1$ at ∞, *if and only if the function* P *has a pole of order* q *at* ∞, *i.e.*

$$P(z) = \sum_{k=-q}^{\infty} P_k z^{-k} \qquad P_{-q} \neq 0. \tag{2}$$

In case $q=-1$, *i.e. the rank is* 0, *we say that* (1) *has a* regular singular point *at* ∞.

(b) *We call equations* (1) *and* ($\bar{1}$) equivalent at ∞, *if and only if a matrix-valued function* A, *holomorphic at* ∞ *and with* $\det A(\infty)\neq 0$, *can be found such that the substitution*

$$Y(z) = A(z)\overline{Y}(z) \tag{3}$$

carries (1) *into* ($\bar{1}$).

C. **Example.** Consider the 2×2 matrix equation with a regular singular point at ∞, i.e. with $q=-1$:

$$Y'(z) = \left\{\frac{1}{z}\begin{bmatrix} -1, & 0 \\ 0 & 0\end{bmatrix} + \frac{1}{z^2}\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}\right\} Y(z). \tag{4}$$

According to Result A, this is equivalent at $z=\infty$ to an equation

Received by the editors August 9, 1958 and, in revised form, January 5, 1959.

[1] Due to Poincaré and Birkhoff, respectively.

$$\overline{Y}'(z) = \frac{\overline{P}_1}{z}\,\overline{Y}(z), \qquad \overline{P}_1 \neq 0. \tag{4}$$

From (3) it follows that $\overline{P}_1$ is similar to the coefficient of $1/z$ in (4); in fact

$$\overline{P}_1 = A_0^{-1}\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} A_0, \qquad A_0 = A(\infty).$$

Now as z completes a positive circuit about the point ∞, the solution Y of (4) gets post-multiplied by a matrix M, called the *monodromic matrix* of (4) associated with the singularity at ∞. Likewise, $\overline{Y}$ gets post-multiplied by a matrix $\overline{M}$ when z completes a positive circuit of ∞. But since the values of the function $\overline{P}(z) = \overline{P}_1/z$ commute, $\overline{M}$ is similar to $\exp(-2\pi i \overline{P}_1)$,[2] i.e. to

$$A_0^{-1} \exp\left(2\pi i \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) A_0.$$

The matrix within the exponential being idempotent, it follows that $\overline{M} = I$. On the other hand, we find by direct integration that (4) has a matrix solution

$$Y(z) = \begin{bmatrix} (\log z)/z, & 1/z \\ 1 & 0 \end{bmatrix}.$$

Since any matrix solution of (4) will be of the form $Y(z)\cdot C$, where $\det C \neq 0$, it follows that

$$M = C^{-1}\begin{bmatrix} 1 & 0 \\ -2\pi i & 1 \end{bmatrix} C.$$

Thus $M \neq \overline{M}$. This shows that Y and $\overline{Y}$ cannot be connected by an equation of the type (3) in which the function A is (single-valued) holomorphic and invertible at ∞. The result A thus fails for the equation (4).

D. **Remarks.** In 1909 Birkhoff [1] discussed Result A in non-degenerate cases, and stated [**5**, p. 201] that he hoped to deal later with the case in which $q = -1$ and the eigenvalues of the matrix P_{-q} in (2) differ by integers. Example C shows that the result is incorrect precisely in this case. In a nutshell the trouble is that M depends on all Laurent coefficients of P and not just on the coefficient of z^{-1}—the residue, cf. [**6**, §4].

---

[2] Cf. e.g. [**6**, 2.2 (d), 4.2]; the minus sign in this expression arises from the fact that a positive circuit of ∞ is a negative circuit of 0.

In his 1913 paper [3], Birkhoff stated Result A in the general form given above, and gave a complete proof for the case in which M can be diagonalized by a similarity transformation, i.e. when M has simple elementary divisors. We may thus conclude:

The Result A *is valid under the assumption that the monodromic matrix M of* (1) *associated with the singularity at* ∞ *has simple elementary divisors.*

It remains to be investigated to what extent this assumption can be weakened. But the necessity for some such assumption shows that the notion of *equivalence* introduced by Birkhoff is not as useful in reducing the complexity of linear differential systems as has been believed.

In his 1913 paper [3] Birkhoff also gave the outlines of a proof of Result A for the case in which M cannot be diagonalized [5, pp. 256–257]. If a proof along these lines were possible, Example C would also disprove the following deep matrix-function-theoretic result on which Birkhoff's entire argument rests, and which he uses in his other work, e.g. in [4]:

BIRKHOFF'S LEMMA [2]. *Every $n\times n$ matrix-valued function F, which is holomorphic and invertible on $[|z|\geqq 1]$ admits the factorization*

$$F(z) = F_+(z)F_-(z)R(z)$$

where F_+, F_- are holomorphic and invertible on $[|z| < \infty]$ and $[1<|z|\leqq \infty]$ respectively, and $R(z)$ is a diagonal matrix with entries $z^{k_1}, \cdots, z^{k_n}$, in which $k_1, \cdots, k_n$ are integers.

This lemma seems to be correct, however. At any rate the writer's attempts to construct a proof of Result A in the degenerate case along the lines suggested by Birkhoff have failed.

REFERENCES

1. G. D. Birkhoff, *Singular points of ordinary linear differential equations*, Trans. Amer. Math. Soc. vol. 10 (1909) pp. 436–470.

2. ———, *A theorem on matrices of analytic functions*. Math. Ann. vol. 74 (1913) pp. 122–133.

3. ———, *Equivalent singular points of ordinary linear differential equations*, Math. Ann. vol. 74 (1913) pp. 134–139.

4. ———, *Infinite products of analytic matrices*, Trans. Amer. Math. Soc. vol. 17 (1916) pp. 386–404.

5. ———, *Collected mathematical papers*, vol. I, American Mathematical Society, New York, 1950.

6. P. Masani, *The rational approximation of operator-valued functions*, Proc. London Math. Soc. (3) vol. 6 (1956) pp. 43–58.

THE INSTITUTE OF SCIENCE, BOMBAY 1, INDIA AND
UNIVERSITY OF CHICAGO