SOME HILBERT SPACES OF ENTIRE FUNCTIONS

LOUIS DE BRANGES

We study here the structure of Hilbert spaces whose elements are entire functions and which have the following three properties.

(H1) Whenever $f(z)$ is in the Hilbert space and w is a nonreal zero of $f(z)$, the function $f(z)(z-\bar{w})/(z-w)$ is in the Hilbert space and has the same norm.

(H2) Whenever w is any nonreal complex number, the linear functional defined on the Hilbert space by $f(z)\to f(w)$, which gives each function in the Hilbert space its value at w, is continuous.

(H3) Whenever $f(z)$ is in the Hilbert space, the function $f^*(z)=\bar{f}(\bar{z})$ is in the Hilbert space and has the same norm.

We are able to give a fairly complete discussion of the structure of such spaces. We would like to pose the problem of structure for Banach spaces whose elements are entire functions and which satisfy H1, H2, and H3 with the word "Hilbert space" replaced by "Banach space."

The reader who is interested in linear transformations in Hilbert space may want to consider a linear transformation, "multiplication by z", defined by $f(z)\to zf(z)$ whenever $f(z)$ and $zf(z)$ are in the Hilbert space. The hypothesis H2 implies that the transformation has a closed graph. The hypothesis H1 implies that the transformation is symmetric and has deficiency index (1, 1). The hypothesis H3 provides a conjugation with respect to which the transformation is real. We will not go into the Hilbert space interpretation now, but the reader who is interested in it should be able to fill in some of the details by reference to Stone [6].

We use the letter $\mathcal{H}$ to stand for a Hilbert space whose elements are entire functions satisfying H1 and H2, and usually H3. If $f(z)$ is in the Hilbert space, we write the norm $\|f(t)\|$ as if t were a dummy variable of integration. If $f(z)$ and $g(z)$ are in the Hilbert space, the inner product is written $\langle f(t), g(t)\rangle$. For each nonreal complex member w, $K(w, z)$ as a function of complex z is to be the unique element of the Hilbert space such that for each $f(z)$ in the Hilbert space

$$f(w) = \langle f(t), K(w, t)\rangle. \tag{1}$$

The existence and uniqueness of such a $K(w, z)$ follow from H2 and the Riesz representation of a continuous linear functional on a Hilbert space.

Received by the editors October 16, 1958 and, in revised form, December 19, 1958.

THEOREM. *If $\mathcal{H}$ is a Hilbert space of entire functions satisfying* H1, H2, *and* H3 *and containing a nonzero element, there is an entire function $E(z)$ such that for $y>0$, $|E(\bar{z})|<|E(z)|$ and the Hilbert space consists exactly of the entire functions $f(z)$ such that*

$$\|f(t)\|_E^2=\int |f(t)|^2|E(t)|^{-2}dt<\infty \tag{2}$$

and $(z=x+iy)$

$$|f(z)|^2 \leqq (4\pi y)^{-1}\|f(t)\|_E^2(|E(z)|^2-|E(\bar{z})|^2). \tag{3}$$

Furthermore, $E(z)$ can be so chosen that $\| \quad \|_E$ agrees with the Hilbert space norm.

A converse statement also is true and will not be proved. Let $E(z)$ be any entire function such that for $y>0$, $|E(\bar{z})|<|E(z)|$. Let $\mathcal{H}(E)$ be the set of entire functions $f(z)$ satisfying (2) and (3). Then, with $\| \quad \|_E$ as norm, $\mathcal{H}(E)$ is a Hilbert space of entire functions satisfying H1, H2, and H3. If $f(z)$ is in $\mathcal{H}(E)$, then (integrating on the real axis)

$$\frac{f(z)}{E(z)}=\frac{1}{2\pi i}\int\frac{f(t)dt}{E(t)(t-z)} \qquad (y>0),$$

$$0=\frac{1}{2\pi i}\int\frac{f(t)dt}{E(t)(t-z)} \qquad (y<0).$$

All this information about $\mathcal{H}(E)$ is submitted without proof and is used in the proof of the theorem. We refer to the introduction to Paley and Wiener [5] where similar versions of Cauchy's formula are discussed.

LEMMA 1. *Let $\mathcal{H}$ be a Hilbert space of entire functions satisfying* H1 *and* H2. *For nonreal complex numbers z and w, $K(z, w)=\overline{K}(w, z)$ and $K(w, w)\geqq 0$. If the Hilbert space contains a nonzero element, the strict inequality always holds.*

LEMMA 2. *If $\mathcal{H}$ is a Hilbert space of entire functions satisfying* H1, H2, *and* H3, *and if w and z are not real, then $K(\bar{w}, z)=K(\bar{z}, w)$.*

LEMMA 3. *If $\mathcal{H}$ is a Hilbert space satisfying* H1 *and* H2 *and if w_1, w_2, w_3, w_4, and z are nonreal complex numbers and if*

$$L(w, z)=(\bar{w}-z)K(w, z),$$

then

$$\begin{aligned}&L(w_1, w_3)L(w_2, z)L(\bar{z}, w_4)\\&-L(w_2, w_3)L(w_1, z)L(\bar{z}, w_4)\\&-L(w_1, w_4)L(w_2, z)L(\bar{z}, w_3)\\&+L(w_2, w_4)L(w_1, z)L(\bar{z}, w_3) = 0.\end{aligned}$$

LEMMA 4. *If $\mathcal{H}$ is a Hilbert space of entire functions which satisfies* H1, H2, *and* H3, *then there is an entire function $E(z)$ such that*

$$K(w, z) = \frac{\bar{E}(w)E(z) - E(\bar{w})E^*(z)}{2\pi i(\bar{w} - z)}.$$

The following notation will be used when we have a Hilbert space of entire functions satisfying H1, H2, and H3. Let $E(z)$ be the choice of an entire function as in Lemma 4. For instance, the entire function might be uniquely specified by requiring that $E(-i)=0$ and $E(i)>0$. For each real number α, let $C(\alpha, z)$ and $S(\alpha, z)$ be the entire functions of z such that

$$e^{i\alpha}E(z) = C(\alpha, z) - i(S\alpha, z) \tag{4}$$

where $C^*(\alpha, z)=C(\alpha, z)$ and $S^*(\alpha, z)=S(\alpha, z)$. We will make some calculations with these functions which are patterned after those of [3]. From the Hilbert space point of view, essentially what we are doing is computing the spectral measures of the self adjoint extensions of the transformation "multiplication by z".

LEMMA 5. *Let $\mathcal{H}$ be a Hilbert space of entire functions satisfying* H1, H2, *and* H3, *and containing a nonzero element. Then, for each real number α, the zeros of $S(\alpha, z)$ and $C(\alpha, z)$ are real and separate each other. In particular, if w is a common zero of $S(\alpha, z)$ and $C(\alpha, z)$, the multiplicities differ by at most* 1. *The functions $S(\alpha, z)(z-w)^{-1}$ where w ranges in the zeros of $S(\alpha, z)$, which are not zeros of $C(\alpha, z)$ or have greater multiplicity than the corresponding zero of $C(\alpha, z)$, are an orthogonal set in the Hilbert space.*

LEMMA 6. *In the situation of Lemma* 5, *if $S(\alpha, z)$ is not in the Hilbert space, the orthogonal set is complete.*

LEMMA 7. *Under the hypotheses of Lemma* 5, *there is at most one real number α modulo π such that $S(\alpha, z)$ is in the Hilbert space.*

PROOF OF LEMMA 1. By appropriate substitutions for $f(z)$ in (1), we have

$$K(z, w) = \langle K(z, t), K(w, t)\rangle.$$

The lemma now follows from the symmetry and positivity properties of an inner product. The only thing that needs verification is that when the Hilbert space contains a nonzero element the inequality is strict. In fact, if we have $K(w, w)=0$, then by the Schwarz inequality applied to (1) we have $f(w)=0$ for every $f(z)$ in the Hilbert space. By H1, $f(z)(z-\bar{w})/(z-w)$ is in the Hilbert space and hence it also vanishes at w. Continuing inductively, for each $m=0, 1, 2, \cdots,$

$$f(z)(z-\bar{w})^m/(z-w)^m$$

is in the Hilbert space and vanishes at w. Since $f(z)$ is analytic, it vanishes identically.

Proof of Lemma 2. By expressing the Hilbert space inner product in terms of the norm, the reader will easily verify that H3 implies that for $f(z)$ and $g(z)$ in the Hilbert space

$$\langle f^*(t), g^*(t)\rangle = \langle g(t), f(t)\rangle.$$

In other words, the star mapping is a conjugation in the sense of Stone [6, pp. 357–360]. By appropriate substitutions in (1), we have

$$K^*(w, z) = \langle K^*(w, t), \quad K(z, t)\rangle = \langle K^*(z, t), \quad K(w, t)\rangle = K^*(z, w).$$

The lemma now follows.

Proof of Lemma 3. The function

$$f(z) = K(w_1, z)K(w_2, z_1) - K(w_2, z)K(w_1, z_1)$$

belongs to the Hilbert space as a function of z and vanishes at z_1. If z_1 is not real, then by H1, the product of this function with

$$(z-\bar{z}_1)/(z-z_1)$$

is in the Hilbert space. Similarly,

$$g(z) = K(w_3, z)K(w_4, \bar{z}_1) - K(w_4, z)K(w_3, \bar{z}_1)$$

and its product by $(z-z_1)/(z-\bar{z}_1)$ are in the Hilbert space. The reader will easily verify from H1, by expressing the Hilbert space norm in terms of the inner product, that

$$\langle f(t)(t-\bar{z}_1)/(t-z_1), g(t)\rangle = \langle f(t), g(t)(t-z_1)/(t-\bar{z}_1)\rangle.$$

The lemma now follows on substituting what $f(z)$ and $g(z)$ are and expanding using (1) and Lemma 1. The computation is long, but has no complications.

Proof of Lemma 4. Write the identity of Lemma 3 with $w_2=i$, $w_4=-i$, and $z=i$. The conclusion of the lemma follows on taking

$$E(z) = \pi^{1/2}K^{-1/2}(i, i)(1-iz)K(i, z)$$

after using Lemma 2 and making a few changes of variable.

PROOF OF LEMMA 5. With the conventions (4), Lemma 4 becomes

$$K(w, z) = \frac{\bar{S}(\alpha, w)C(\alpha, z) - \bar{C}(\alpha, w)S(\alpha, z)}{\pi(\bar{w} - z)}. \tag{5}$$

When $w=z$, the inequality of Lemma 1 implies that the zeros of $S(\alpha, z)$ and $C(\alpha, z)$ are real.

We want to apply the formula (5) when w is real, and that requires interpretation since the left hand side has not been defined in this case. When w is not real and $f(z)$ is in the Hilbert space, we have by the Schwarz inequality applied to (1),

$$|f(w)|^2 < \|f(t)\|^2 K(w, w).$$

By Lemma 4, $K(w, z)$ can be extended continuously. Since the elements of the Hilbert space are entire functions and hence continuous across the real axis, the last inequality shows that H2 remains valid even if real w are allowed. Now define $K(w, z)$ for real w so that (1) holds. Most of our formulas remain valid for real values of the arguments, but in Lemma 1, the strict inequality does not necessarily hold on the real axis.

If we apply (5) when $z=w$ is real, the inequality of Lemma 1 becomes

$$S'(\alpha, z)C(\alpha, z) - C'(\alpha, z)S(\alpha, z) \geqq 0.$$

It follows that the zeros of $S(\alpha, z)$ and $C(\alpha, z)$ separate each other, as the lemma states.

Let w_1 and w_2 be distinct zeros of $S(\alpha, z)$ at which $C(\alpha, z)\neq 0$. The orthogonality of $S(\alpha, z)(z-w_1)^{-1}$ and $S(\alpha, z)(z-w_2)^{-1}$ follows from (5) and the first identity used in the proof of Lemma 1. The modifications of this argument which must be made for the other relevant zeros of $S(\alpha, z)$ are left to the reader.

PROOF OF LEMMA 6. We will prove the lemma by showing that if $f(z)$ is in the Hilbert space and if $f(z)$ is perpendicular to every element of the orthogonal set of Lemma 5, then $f(z)$ is at most a complex multiple of $S(\alpha, z)$, with the implication that if $S(\alpha, z)$ is not in the Hilbert space, then $f(z)$ is identically zero. By applying the Schwarz inequality to (1), we get (with w replaced by z)

$$|f(z)|^2 < \|f(t)\|^2 K(z, z) = \|f(t)\|^2 (4\pi y)^{-1}(|E(z)|^2 - |E^*(z)|^2). \tag{6}$$

We leave it to the reader to verify using (1) and (5) that the orthogonality of $f(z)$ to the set of Lemma 5 is equivalent to the analyticity of $g(z)=f(z)/S(\alpha, z)$. Using (4), the inequality (6) can now be written

$$|g(z)|^2 \leqq \|f(t)\|^2(2\pi y)^{-1}\operatorname{Im}(S(\alpha, z)/C(\alpha, z)).$$

Since, $\operatorname{Im}(S(\alpha, z)/C(\alpha, z))$ is positive and harmonic in the half plane $y>0$ (because of the inequality of Lemma 1), there is a non-negative measure $\mu=\mu\alpha$ on the Borel sets of the real line such that $\int(1+t^2)^{-1}d\mu(t) < \infty$ and some finite $a>0$ such that for $y>0$

$$\operatorname{Im}(S(\alpha, z)/C(\alpha, z)) = \frac{y}{\pi}\int \frac{d\mu(t)}{(t-x)^2+y^2} + ay.$$

(See Loomis and Widder [4].) Since $S^*(\alpha, z)=S(\alpha, z)$ and $C^*(\alpha, z) = C(\alpha, z)$, the last formula is also valid when $y<0$. Therefore,

$$|g(z)|^2 \leqq \|f(t)\|^2 \frac{1}{2\pi^2}\int \frac{d\mu(t)}{(t-x)^2+y^2} + \frac{a}{2\pi}\|f(t)\|^2.$$

Elementary estimates from this formula (compare with [2, pp. 147–148]) show that $g(z)$ has minimal exponential type and is bounded on the imaginary axis. By Boas [1, p. 83], $g(z)$ is a constant.

The lemma now follows.

Proof of Lemma 7. If $S(\alpha, z)$ is in the Hilbert space, then (6) implies that $S(\alpha, iy)=o(E(iy))$ as $y\rightarrow+\infty$. By the proof of Lemma 3 of [3], this situation can occur for at most one real number α modulo π.

Proof of Theorem. Let $f(z)$ be in the Hilbert space $\mathcal{H}$. If α is any real number such that $S(\alpha, z)$ does not belong to the Hilbert space, apply Parseval's formula to the complete orthogonal set of Lemma 5. Using (1) and (5), the formula becomes explicitly

$$\|f(t)\|^2 = 2\pi i \sum \frac{|f(w)|^2}{E(w)E^{*\prime}(w) - E^*(w)E'(w)},$$

where w ranges in the real numbers corresponding to the elements $S(\alpha, z)(z-w)^{-1}$ in the orthogonal set of Lemma 5. (In case the denominator of this fraction vanishes, we see using (1) that the numerator does also, and the fraction in this case is to be defined by continuity.) The proof of the theorem of [3] now applies to show that

$$\|f(t)\|^2 = \int |f(t)|^2|E(t)|^{-2}dt.$$

There is some difficulty caused by the possible real zeros of $E(z)$, but since these form a set of measure zero, it is easily overcome. Note that with this computation of the Hilbert space norm, (3) is just (6).

In other words, the Hilbert space $\mathcal{H}$ is a closed subspace of $\mathcal{H}(E)$. By (1), for all complex members w,

$$f(w) = \int f(t)\overline{K}(w, t) \mid E(t) \mid^{-2} dt. \tag{7}$$

The reader will easily verify that the same formula is also valid for every function $f(z)$ in $\mathcal{H}(E)$, and is a consequence of the two integral formulas stated just after the definition of $\mathcal{H}(E)$.

The problem now is to show that $\mathcal{H}$ fills $\mathcal{H}(E)$. Let $f(z)$ be any element of $\mathcal{H}(E)$ and let $g(z)$ be the projection of $f(z)$ into $\mathcal{H}$. Then, by two uses of (7) we have for each complex number w,

$$f(w) = \int f(t)\overline{K}(w, t) \mid E(t) \mid^{-2} dt = \int g(t)\overline{K}(w, t) \mid^{-2} dt = g(w).$$

Therefore, $f(z) = g(z)$ is in $\mathcal{H}$. Q.E.D.

REFERENCES

1. R. P. Boas, Jr., *Entire functions*, New York, Academic Press, 1954.

2. L. de Branges, *Local operators on Fourier transforms*, Duke Math. J. vol. 25 (1958) pp. 143–154.

3. ———, *Some mean squares of entire functions*, Proc. Amer. Math. Soc. vol. 10 (1959) pp. 833–839.

4. L. H. Loomis and D. V. Widder, *The Poisson integral representation of functions which are positive and harmonic in a half plane*, Duke Math. J. vol. 9 (1942) pp. 643–645.

5. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 19, 1934, 184 pp.

6. M. H. Stone, *Linear transformations in Hilbert space and their applications to analysis*, Amer. Math. Soc. Colloquium Publications, vol. 15, 1932, 622 pp.

LAFAYETTE COLLEGE