SUMS OF NORMAL ENDOMORPHISMS

R. H. BRUCK[1]

1. **Introduction.** The main object of this note is to prove the following:

THEOREM. *Let* $(L, +)$ *be the additive loop generated by the set of all normal endomorphisms of a loop* G. *A necessary and sufficient condition that* $(L, +, \cdot)$ *be an (ordinary, associative) ring is that* G *be power-associative.*

It should be noted that the definition (introduced in [1]) of a normal endomorphism of a loop G is radically different from the usual definition for groups. Nevertheless (as shown in [2]) the two definitions are equivalent when G is a group. In particular, the present theorem generalizes one of Heerema [3]. We may add that in [2], by assuming that the loop G was Moufang, we were able to be much more explicit than at present about the properties of the ring $(L, +, \cdot)$. For example, if G is merely power-associative and if θ is an element of L which happens to be an endomorphism, we can answer neither of the following questions: (i) Is the complement $1-\theta$ an endomorphism? (ii) Is θ semi-normal in the sense of [2]?

The definition of a normal endomorphism involves the concept of a purely nonabelian (p.n.a.) loop word; this will be discussed in §2. Roughly speaking, p.n.a. loop words are the analogues for loops of the *higher commutator forms* introduced by Philip Hall in the theory of groups, although an inductive definition of p.n.a. loop words as "higher commutator-associator forms" does not seem profitable at present. In particular, the following simple lemma, on which the proof of the theorem hinges, could be generalized to deal with the various terms of the lower central series or derived series of a loop:

LEMMA 1. *Let* H *be a loop generated by a (nonempty) subset* A. *Then the following condition (C) is both necessary and sufficient in order that* H *be an abelian group:*

(C) $W_n(a_1, \cdots, a_n) = 1$ *for every positive integer* n, *every normalized p.n.a. loop word* W_n, *and all choices of* $a_1, \cdots, a_n$ *in* A.

Indeed, our proof of Lemma 1 could easily be rearranged to prove a

Presented to the Society, January 22, 1959; received by the editors November 14, 1958.

[1] This research was supported by a grant from the National Science Foundation.

stronger statement: *The elements* $W_n(a_1, \cdots, a_n)$ *generate the commutator-associator subloop*, H', *of* H.

2. **Preliminaries.** By a loop word W_n we mean an element of the free loop F on n free generators $X_1, \cdots, X_n$. If G is a loop and $a_1, \cdots, a_n$ are elements of G, $W_n(a_1, \cdots, a_n)$ denotes the image of W_n under the (uniquely defined) homomorphism of F into G which maps X_i into a_i for $i=1, 2, \cdots, n$.

As shown in [2], each of the following properties of a loop word W_n implies the others:

(i) W_n *is an element of the commutator-associator subloop* F'.

(ii) W_n *vanishes on every abelian group.*

(iii) *If* G *is a loop with centre* Z *then*

$$W_n(x_1c_1, \cdots, x_nc_n) = W_n(x_1, \cdots, x_n)$$

for all x_i *in* G *and* c_i *in* Z.

A loop word W_n is called *purely nonabelian* (p.n.a.) if it satisfies the equivalent properties (i)–(iii).

Now we need some new definitions. Fix attention on a generator X_i of the free loop F and let θ_i be the (idempotent) endomorphism of F which maps X_i into the identity element, 1, of F and fixes X_j for each $j\neq i$. Then, if W_n is in F, we call X_i a *nonessential argument* of W_n if $W_n\theta_i=W_n$, and an *essential argument* of W_n if $W_n\theta_i\neq W_n$. Note that a loop word is an element of the free subloop on its essential arguments.

A loop word W_n is called *normalized* provided that $W_n\theta_i=1$ for $i=1, 2, \cdots, n$. We shall need the following refinement: W_n is called *essentially normalized* if $W_n\theta_i=1$ whenever X_i is an essential arument of W_n. In other words, W_n is essentially normalized if it is normalized when considered as an element of the free loop on its essential arguments.

LEMMA 2. *If* W_n *is a p.n.a. loop word, there exists a finite set* $\mathfrak{S}_n$ *of essentially normalized p.n.a. loop words such that* W_n *is in the set* $\mathfrak{S}_n^*$ *defined inductively as follows*:

(i) $\mathfrak{S}_n\subset\mathfrak{S}_n^*$.

(ii) *If* A_n, B_n *are in* $\mathfrak{S}_n^*$, *then* A_nB_n *is in* $\mathfrak{S}_n^*$.

PROOF. First we note that, if $n=1$, every loop word W_1 is normalized and hence essentially normalized. Therefore, if $n=1$, we may define $\mathfrak{S}_1$ to consist of the given p.n.a. word W_1.

Next we consider the case $n>1$ and assume inductively that the lemma is true for $n-1$. We define loop words A_n, B_n by

$$A_n = W_n\theta_n, \qquad W_n = A_nB_n.$$

We observe first that A_n, B_n are p.n.a. loop words. In addition, X_n is a nonessential argument of A_n; consequently A_n may be considered as an element A_{n-1} of the free loop on $X_1, \cdots, X_{n-1}$. Therefore, by our inductive assumption, A_n may be built up in the manner of the lemma from a finite set of essentially normalized p.n.a. words each of which has X_n as a nonessential argument. On the other hand, $B_n\theta_n=1$, so B_n is "normalized in X_n." Next we define loop words C_n, D_n by

$$C_n = B_n\theta_{n-1}, \qquad B_n = C_nD_n.$$

Again, C_n, D_n are p.n.a. Moreover, the inductive assumption applies to C_n, while D_n is normalized both in X_n and in X_{n-1}. If $n>2$, we proceed in the same manner to "split" D_n by means of θ_{n-2}. Since the indicated process must come to an end in a finite number of steps, the truth of Lemma 2 is now clear.

Next we must consider the set M of all single-valued mappings of a multiplicative loop G into itself. The product, $\alpha\beta$, and the sum, $\alpha+\beta$, of two elements α, β of M are defined by

$$x(\alpha\beta) = (x\alpha)\beta, \qquad x(\alpha + \beta) = (x\alpha)(x\beta) \tag{2.1}$$

for all x in G. We note that $(M, +)$ is a loop. In addition, if $(L, +)$ is a subloop of $(M, +)$ and x is any fixed element of G, the mapping $\alpha\rightarrow x\alpha$ induces a homomorphism of $(L, +)$ upon a subloop xL of G. As one consequence, if W_n is a loop word,

$$xW_n(\alpha_1, \cdots, \alpha_n) = W_n(x\alpha_1, \cdots, x\alpha_n) \tag{2.2}$$

for all x in G and $\alpha_1, \cdots, \alpha_n$ in $(M, +)$. Moreover

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma \tag{2.3}$$

for all α, β, γ in M, and

$$(\beta + \gamma)\theta = \beta\theta + \gamma\theta \tag{2.4}$$

for all β, γ in M precisely when θ is an endomorphism of G.

And, finally, we must recall the definition of a normal endomorphism. An endomorphism θ of a loop G is *normal* provided that

$$W_n(x_1, \cdots, x_n)\theta = W_n(x_1\theta, x_2, \cdots, x_n) \tag{2.5}$$

for every choice of a positive integer n, a normalized p.n.a. word W_n, and elements $x_1, \cdots, x_n$ of G. When θ is normal, (2.5) also holds with the right hand side replaced by any one of

$$W_n(x_1, x_2\theta, \cdots, x_n), \cdots, W_n(x_1, x_2, \cdots, x_n\theta).$$

3. **Proof of Lemma 1.** Let H be a loop generated by a subset A. If H is an abelian group, condition (C) obviously holds for every p.n.a. loop word W_n. On the other hand, if (C) holds for every normalized p.n.a. loop word W_n, then, by Lemma 2 and the definition of "essentially normalized," (C) also holds for every p.n.a. loop word W_n (not necessarily normalized).

We shall use this strengthened form of (C) to show that $H'=1$, where G' denotes the commutator-associator subloop of a loop G. If h is an element of H', there exists a finite set $a_1, \cdots, a_n$ of elements of A such that h is in K' where K is the subloop generated by $a_1, \cdots, a_n$. Let F be the free loop on $X_1, \cdots, X_n$, and let θ be the homomorphism such that $X_i\theta=a_i$ for $i=1, 2, \cdots, n$. Then θ maps F upon K and F' upon K'. Moreover, by definition, an element W_n of F is in F' precisely when W_n is p.n.a. Hence $K'=1$, so that $h=1$ and therefore $H'=1$. That is, H is an abelian group. This completes the proof of Lemma 1.

4. **Proof of the theorem.** Let G be a loop, let A be the set of all normal endomorphisms of G and let $(L, +)$ be the additive loop generated by A. We note that $AA\subset A$ and that the identity mapping, I, of G is in A. The zero mapping, 0, defined by

$$x0 = 1, \qquad \textit{all } x \textit{ in } G, \tag{4.1}$$

is also in A and is in fact the additive identity element of L. Now let x be any element of G, W_n be any normalized p.n.a. loop word, and $\phi_1, \cdots, \phi_n$ be any n elements of A. Then

$$xW_n(\phi_1, \cdots, \phi_n) = W_n(x\phi_1, \cdots, x\phi_n), \tag{4.2}$$

by (2.2), and

$$W_n(x\phi_1, \cdots, x\phi_n) = W_n(x, \cdots, x)\phi_1 \cdots \phi_n \tag{4.3}$$

by the normality of $\phi_1, \cdots, \phi_n$.

If G is power-associative then, by Lemma 1 applied to the subloop generated by x, the right hand side of (4.3) reduces to $1\phi_1 \cdots \phi_n=1$. Accordingly, (4.2) yields

$$W_n(\phi_1, \cdots, \phi_n) = 0 \tag{4.4}$$

for all $\phi_1, \cdots, \phi_n$ in A. But then, by Lemma 1 applied to the additive loop $(L, +)$, $(L, +)$ is an abelian group. Conversely, if $(L, +)$ is an abelian group, (4.4) holds in particular when each ϕ_i is I. In this case, (4.2) yields

$$W_n(x, \cdots, x) = 1$$

for every x. Therefore each element of G generates an abelian group.

We have now shown that $(L, +)$ is an abelian group precisely when G is power-associative. In particular, then, G must be power-associative if $(L, +, \cdot)$ is a ring.

We assume henceforth that $(L, +)$ is an abelian group.

Since $AA \subset A$, two simple arguments based on (2.4), (2.3) allow us to prove, respectively, that $LA \subset L$ and that $LL \subset L$. (For details see [2].) Hence $(L, \cdot)$ is a semigroup. In addition, for each fixed pair α, β of elements of L, the mapping

$$\gamma \to (\alpha + \beta)\gamma - \alpha\gamma - \beta\gamma$$

is an endomorphism of $(L, +)$ which sends A into 0. Hence

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma \tag{4.5}$$

for all α, β, γ in L. In view of (2.3), (4.5), $(L, +, \cdot)$ is clearly a ring. This completes the proof of the theorem.

References

1. R. H. Bruck, *A survey of binary systems*, Ergebnisse der Mathematik und Ihrer Grenzgebiete, Neue Folge, Heft 20, Berlin-Göttingen-Heidelberg, Springer-Verlag, 1958; Math. Rev. vol. 20 (1959) p. 13.

2. ———, *Normal endomorphisms*, Illinois J. Math. to appear.

3. Nicholas Heerema, *Sums of normal endomorphisms*, Trans. Amer. Math. Soc. (1957) pp. 137–143; Math. Rev. vol. 18, p. 559.

University of Wisconsin