AN EXTREMAL PROBLEM FOR POLYNOMIALS

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Problem. Consider the class of nth order polynomials \( \{ f(z) \} \) such that \( f(1) = 0 \), \( |f(z)| \leq 1 \) for \( |z| = 1 \). From this class select that polynomial for which

\[
\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \text{ is greatest.}
\]

For the solution we require the following

Lemma. Let \( h(z) = \sum_{-N}^{N} h_n z^n \), (\( h_n = h_{-n} \)). Then there exists a polynomial \( f(z) \) of degree \( N \) such that \( h(z) = |f(z)|^2 \) for \( |z| = 1 \) if and only if \( h(z) \geq 0 \) for \( |z| = 1 \). Proof is available in reference [1].

The function \( 1 - |f(z)|^2 \) (with \( \bar{z} \) replaced by \( 1/z \)) satisfies the conditions of the lemma for any \( f(z) \) that satisfies the conditions of the problem. Thus, we can write, \( 1 - |f(z)|^2 = |g(z)|^2 \), where \( g(z) \) satisfies the conditions that \( |g(z)| \leq 1 \) for \( |z| = 1 \) and \( |g(1)| = 1 \). (Without real loss of generality, we take this last to mean \( g(1) = 1 \).)

In addition, for \( f(z) \) to solve the problem, the associated \( g(z) \) must minimize the integral

\[
\frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^2 d\theta.
\]

Writing \( g(z) = \sum_{0}^{N} g_n z^n \), we see that we are seeking to minimize the quantity \( \sum_{0}^{N} |g_n|^2 \) subject to the constraint that \( \sum_{0}^{N} g_n = 1 \). A straightforward application of the Schwarz Inequality yields:

\[
1 = \sum_{0}^{N} g_n \leq \left( \sum_{0}^{N} |g_n|^2 \right)^{1/2} (N+1)^{1/2}.
\]

The sum-of-squares is smallest when we set \( g_n = 1/(N+1) \), and obtain for the corresponding \( g(z) \),

Received by the editors November 21, 1958 and, in revised form, January 15, 1959.
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\[ g(z) = \frac{1}{N + 1} \sum_{n=0}^{N} z^n. \]

This well known function is equal to unity for \( z = 1 \) and vanishes when \( z \) is any of the other \( N+1 \)st roots of unity. It is easy to show that

\[ |g(e^{i\theta})| = \left| \frac{\sin \frac{N + 1}{2} \theta}{(N + 1) \sin \theta/2} \right|, \quad -\pi \leq \theta \leq \pi. \]

From this it is seen that \( |g(z)| \leq 1 \) for \( |z| = 1 \), so that this constraint is satisfied even though we did not impose it in determining \( g(z) \). As a consequence, the function \( 1 - |g(z)|^2 \) (with 1/z set for \( \bar{z} \)) satisfies the conditions of the lemma so that the associated function, \( f(z) \), is the solution to the problem.

One obtains a fair idea of the nature of \( f(z) \) from the observations that it never passes outside the unit circle, it passes through the origin for \( z = 1 \), and it is tangent on the inside to the unit circle at all the other \( N+1 \)st roots of unity. The value of the integral being maximized is \( N/(N+1) \).

The method of computation of the \( N \)th order \( f \) is given essentially in the proof of the lemma:

Solve the reciprocal equation \( 1 - g(z)g(1/z) = 0 \), and, from each pair of reciprocal roots select one member. Then \( f \) is that \( N \)th order polynomial having these selected quantities for roots.

\( f \) must also be properly normalized, of course. To remove an irrelevant ambiguity, we may specify that none of the roots should lie outside the unit circle.

For example:

\[ f_0 = 0, \]
\[ f_1 = (z - 1)/2, \]
\[ f_2 = ((1 + 3^{1/2})z^2 - 2z + (1 - 3^{1/2}))/(18)^{1/2}. \]

The author is grateful to the referee for the following reference.

**Reference**


**General Atronics Corporation, Bala Cynwyd, Pennsylvania**