AN OSCILLATION THEOREM FOR SOLUTIONS OF A CLASS OF PARTIAL DIFFERENCE EQUATIONS

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In 1908 Hadamard [2] raised the question: if \( u \) is the solution of

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 u = p
\]

in a region \( R \) with \( u = \partial u / \partial n = 0 \) on the boundary of \( R \), does the restriction \( p \geq 0 \) imply that \( u \geq 0 ? \) Duffin, [1], showed that it is possible, when \( R \) is an infinite strip, to construct solutions for this problem which oscillate in sign infinitely often, where \( p \) is non-negative everywhere and zero outside a finite subset of \( R \). Loewner [4] and Szegö [5] have constructed other examples showing this same kind of behavior. In this paper we consider a finite difference analogue of Duffin’s problem, but prove results analogous to his for a general class of partial difference equations, of which the biharmonic equation is a special case.

Let \( u(x, y) \) satisfy a linear partial difference equation of the form:

\[
(1) \quad Lu(x, y) = \sum_{i=-r}^{r} \sum_{j=-m}^{m} c_{ij}u(x+i, y+j) = 0
\]

for all integral \((x, y)\) in the strip \( S_N: 0 \leq y \leq N, x \geq 0 \). The boundary values of \( u \) for \(-m \leq y < 0\) and for \( N < y \leq N+m \) are assumed to be zero, and those for \(-\nu \leq x < 0, 0 \leq y \leq N\) are arbitrary real numbers. The coefficients \( c_{ij} \) are assumed to be real constants, and \( c_{ij} = c_{i,-j} \). Let

\[
\begin{vmatrix}
  c_{r,0} & c_{r,1} & \cdots & c_{r,N} \\
  c_{r,-1} & c_{r,0} & \cdots & c_{r,N-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{r,-N} & c_{r,-N+1} & \cdots & c_{r,0}
\end{vmatrix} \neq 0,
\]

where \( c_{ij} = 0 \) if \(|i| > \nu \) or \(|j| > m \). Then the solution \( u(x, y) \) is uniquely determined by its boundary values and by its values for \( 0 \leq x \leq \nu - 1, 0 \leq y \leq N \). These values are assumed to be real, and \( u(x, y) \) is then real for all \((x, y)\) in the strip \( S_N \).

Let \( Eu(x, y) = u(x+1, y), \; Q_j(E) = \sum_{i=-r}^{r} c_{ij}E_i \), and let the partial
difference equation (1) be thought of as a system of ordinary difference equations in \(u(x, 0), u(x, 1), \cdots, u(x, N)\). We then have, for each integer \(y\) in \([0, N]\),

\[
\Delta(E)u(x, y) = 0, \quad x \geq 0,
\]

where

\[
\Delta(E) = \begin{vmatrix}
Q_0(E) & Q_1(E) & \cdots & Q_N(E) \\
Q_{-1}(E) & Q_0(E) & \cdots & Q_{N-1}(E) \\
\vdots & \vdots & \ddots & \vdots \\
Q_{-N}(E) & Q_{-N+1}(E) & \cdots & Q_0(E)
\end{vmatrix}
\]

Let the distinct roots of the equation \(\Delta(z) = 0\) be \(z = r_1, r_2, \cdots, r_K\), and let these roots have multiplicities \(1+s_1, 1+s_2, \cdots, 1+s_K\) respectively. We shall refer to these roots as the characteristic roots of the operator \(L\) for the strip \(S_y\). Then

\[
u(x, y) = \sum_{\rho=1}^{K} \sum_{\sigma=0}^{s_\rho} a_{\rho\sigma}x^{\sigma}r_\rho^y,
\]

where the coefficients \(a_{\rho\sigma}\) are functions of \(y\).

We shall be particularly concerned with operators \(L\) whose characteristic roots are all nonreal and take up first a sufficient condition for an operator to have this property. Let

\[
f(r, \theta) = \sum_{j=-N}^{N} Q_j(r)e^{ij\theta} = Q_0(r) + 2 \sum_{j=1}^{N} Q_j(r) \cos j\theta.
\]

**Theorem 1.** If, for every real number \(r\), \(f(r, \theta)\) is non-negative and not identically zero for \(0 \leq \theta \leq 2\pi\), then the characteristic roots of the operator \(L\) are all nonreal.

**Proof.** Let \(D(r, \xi)\) be the Toeplitz form corresponding to \(f(r, \theta)\),

\[
D(r, \xi) = \frac{1}{2\pi} \int_{0}^{2\pi} f(r, \theta) \left| T(\xi, \theta) \right|^2 d\theta = \sum_{i=0}^{N} \sum_{j=0}^{N} Q_{i-j}(r) \xi_i \xi_j,
\]

where \(T(\xi, \theta) = \sum_{k=0}^{N} \xi_k e^{ik\theta}, \xi_k\) real. Then, for any real \(r\), this quadratic form is positive definite, and therefore its determinant cannot be zero, i.e.: \(\Delta(r) \neq 0\).

**Theorem 2.** Let \(L_1\) and \(L_2\) be two operators of the form defined by (1), and let \(f_1(r, \theta)\) and \(f_2(r, \theta)\) be the corresponding functions defined by (4). Then the function defined by (4) for the operator \(L_1L_2\) is \(f_1f_2\), provided \(N\) is sufficiently large.
Proof. Let
\[ L_1(u, y) = \sum_{j=-m_1}^{m_1} Q_j^{(1)}(E)u(x, y + j), \]
\[ L_2u(x, y) = \sum_{j=-m_2}^{m_2} Q_j^{(2)}(E)u(x, y + j), \]
and let \( Q_j^{(1)} = 0 \) for \(|j| > m_1\), \( Q_j^{(2)} = 0 \) for \(|j| > m_2\). Then
\[ L_1L_2u(x, y) = \sum_{j=-m_1}^{m_1} \sum_{k=-m_2}^{m_2} Q_j^{(1)}(E)Q_k^{(2)}(E)u(x, y + k + j) \]
\[ = \sum_{l=-m_1+m_2}^{m_1+m_2} Q_l(E)u(x, y + l), \]
where \( Q_l(E) = \sum_{j+k=l} Q_j^{(1)}Q_k^{(2)} \). Hence, if \( N \geq m_1+m_2 \), the function \( f(r, \theta) \) defined by (4) for the operator \( L_1L_2 \) is given by
\[ f(r, \theta) = \sum_{l=-\infty}^{\infty} Q_l(r)e^{il\theta} = \sum_{l=-\infty}^{\infty} \sum_{j+k=l} Q_j^{(1)}Q_k^{(2)}e^{il\theta} = f_1f_2. \]

The usual difference operator, \( L^{(H)} \), arising from Laplace's equation is defined by
\[ L^{(H)}u(x, y) = \sum_{j=-1}^{1} Q_j^{(H)}(E)u(x, y + j), \]
where \( Q_0^{(H)} = E^{-1} - 4 + E, \ Q_1^{(H)} = Q_{-1}^{(H)} = 1 \). The usual operator \( L^{(B)} \) arising from the biharmonic equation is taken equal to \( L^{(H)^2} \), and
\[ L^{(B)}u(x, y) = \sum_{j=-2}^{2} Q_j^{(B)}(E)u(x, y + j), \]
where \( Q_0^{B} = E^{-2} - 8E^{-1} + 20 - 8E + E^2, \ Q_1^{B} = Q_{-1}^{B} = 2E^{-1} - 8 + 2E, \ Q_2^{B} = Q_{-2}^{B} = 1 \).

Theorem 3. The characteristic roots of the biharmonic operator are nonreal for all values of \( N \).

Proof. For \( N \geq 2 \), the result follows from Theorems 1 and 2 and the fact that the biharmonic operator is the square of the harmonic operator. For \( N = 0, 1 \) the determinants \( \Delta(r) \) defined by (2) become
\[ \Delta_0(r) = \rho^2 - 8\rho + 18 \]
and
\[ \Delta_1(r) = (\rho^2 - 8\rho + 18)^2 - (2\rho - 8)^2 = (\rho^2 - 6\rho + 10)(\rho^2 - 10\rho + 26) \]
respectively, where \( \rho = r + 1/r \geq 2 \), and it is obvious that \( \Delta_0 > 0, \Delta_1 > 0 \) for real \( r \).

Let us assume that the characteristic roots of an operator \( L \) for the strip \( S_N \) are all nonreal. For any fixed integer \( y \) in \([0, N]\) let any real solution (3) of (1) be written in the form

\[
(6) \quad u(x, y) = \sum_{a=1}^{l} \rho_a (q_a e^{i\theta_a} + \bar{q}_a e^{-i\theta_a}),
\]

where \( \rho_1 \geq \rho_2 \geq \cdots \rho_l > 0 \), and where, for each \( \alpha \), \( 0 < \theta_a < \pi \), and \( q_a \) is a polynomial in \( x \) not identically zero. Let \( k \geq 1 \) be the integer such that \( \rho_1 = \rho_2 = \cdots = \rho_k \) and either \( k = l \) or \( \rho_k > \rho_{k+1} \). Then

\[
(7) \quad u(x, y) = \rho_1 x^\mu \sum_{a=1}^{k} (c_a e^{i\theta_a} + \bar{c}_a e^{-i\theta_a}) + o(1),
\]

where \( c_1, c_2, \cdots, c_k \) are constants, not all zero.

**Theorem 4.** If the characteristic roots of the operator \( L \) in the strip \( S_N \) are all nonreal, then the solution of (1), for each \( y \) in \([0, N]\), will change sign infinitely many times as \( x \to \infty \).

**Proof.** For any fixed \( y \) in \([0, N]\) let \( u(x, y) \) be expressed by (7). Let

\[
T(x) = \sum_{a=1}^{k} (c_a e^{i\theta_a} + \bar{c}_a e^{-i\theta_a}).
\]

For any \( \epsilon > 0 \) let \( I_\epsilon \) be the set of positive integers \( \nu \) for which there exist corresponding integers \( \nu_\alpha \) such that

\[
| \nu \theta_\alpha - 2\pi \nu_\alpha | < \epsilon, \quad \alpha = 1, 2, \cdots, k.
\]

For any \( \epsilon > 0 \), \( I_\epsilon \) contains infinitely many members, (see [3, p. 169]), and if \( \nu \in I_\epsilon \),

\[
(8) \quad | T(x) - T(x + \nu) | < C\epsilon, \quad C = 2 \sum_{a=1}^{k} | c_a |.
\]

For any positive integer \( A \), let us suppose that there are no sign changes in \( T(x) \) for \( x \geq A \), say \( T(x) \geq 0 \) for \( x \geq A \). Choose \( x_0 \geq A \) such that \( T(x_0) > 0 \), and let \( \epsilon = (1/2C)T(x_0) \). Then \( T(x) > 2^{-1}T(x_0) \) for
\( x = x_0 + \nu, \nu \in I_x \) and \( \sum_{x = A}^{M} T(x) \to \infty \) as \( M \to \infty \). This, however, is a contradiction, since

\[
\left| \sum_{x = A}^{M} T(x) \right| \leq \frac{2C}{1 - e^{i\theta}}, \quad \theta = \min \{ \theta_x \},
\]

for all \( M \). The assumption \( T(x) \leq 0 \) for \( x \geq A \) also leads to a contradiction. Hence, for any \( A \), \( T(x) \) must change sign at least once, and consequently an infinite number of times, for \( x \geq A \).

Let \( T(x_0) > 0, \varepsilon = (1/2C)T(x_0) \). Then, from (7), \( u(x_0 + \nu, y) > 0 \) for all sufficiently large values of \( \nu \) in \( I_x \). In the same way we show that there must exist infinitely many values of \( x \) such that \( u(x, y) < 0 \).

Returning to Hadamard's question stated in the first paragraph, let us assume that \( u \) satisfies the partial difference equation \( Lu = p \) in the doubly infinite strip \( 0 \leq y \leq N \) and that \( u = 0 \) for \( -m \leq y < 0 \) and \( N < y \leq N + m \). If the characteristic roots of \( L \) in the strip \( S_N \) are nonreal and if \( p(x, y) = 0 \) for \( x \geq 0 \), then, by Theorem 4, \( u(x, y) \) oscillates in sign infinitely often as \( x \to \infty \). This is true regardless of the values of \( p(x, y) \) for \( x < 0 \).

**References**


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