A NOTE ON PROJECTIVE RESOLUTIONS

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A. M. Gleason has proved the existence of essentially unique minimal projective resolutions in the category of compact spaces [2]. Gleason's argument involves a chain of lemmas about closed sets and culminates in a point-by-point construction of the required projective object. This procedure has the advantage of explicitly identifying the projective objects, which turn out to be the extremally disconnected spaces.[1] However, it may be of interest to give a shorter proof which shows more clearly how the special properties of compact spaces are used.

The topological kernel of the proof lies in the following two lemmas.

LEMMA 1. *In Hausdorff topological spaces, if $t: P \to P$ is not the identity, then there is a proper closed subspace S such that $S \cup t^{-1}(S) = P$.*

PROOF. There is a point p such that $t(p) \neq p$. Then p and $t(p)$ have disjoint open neighborhoods, U, V. Let S be the complement of $U \cap t^{-1}(V)$.

LEMMA 2 [2, 2.4]. *In compact spaces, if $f: P \to X$ is onto, then P has a closed subspace S which is minimal subject to $f(S) = X$.*

As Gleason observes, this is a well known consequence of Zorn's lemma. We need one more lemma, which is valid in many other categories also. Let us call a compact space *free* if it is the Stone-Čech compactification βD of a discrete space D. Recall Gleason's (essentially standard) definition of a *projective* object X: an X such that whenever $q: Y \to Z$ is onto, any $f: X \to Z$ can be lifted, i.e., there is $g: X \to Y$ such that $qg = f$.

LEMMA 3. *In compact spaces, every object is a quotient of a free object and every free object is projective.*

PROOF. For each X, let D be a discrete space in one-to-one correspondence with X by $p: D \to X$; then p has a unique continuous

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[1] Probably the shortest proof of this is the direct proof in [2]. If we use Stone's theory of representations for Boolean algebras, the result follows from the corollary to Lemma 3 below in view of Sikorski's characterization of the injective Boolean algebras. Gleason [2] gives the relevant material from Sikorski and Stone, except that reference to [3] is omitted; Theorem 4.7 of [3] is stated and proved in [2, pp. 485–486].

extension $q: \beta D \to X$ [1]. Second, for any $f: \beta D \to Z$ and any onto mapping $h: Y \to Z$, define $e: D \to Y$ by assigning to each point y of D some point of $h^{-1}(f(y))$; again e has a unique continuous extension $g: \beta D \to Y$. Since hg and f agree on the dense subset D of βD, they agree everywhere, and βD is projective.

COROLLARY. *The projective objects are precisely the retracts of the free objects.*

PROOF. Suppose P is projective. There is an onto mapping $q: \beta D \to P$ for some free βD, and the identity on P can be lifted to an embedding of P in βD, so that q becomes a retraction. Conversely, if R is a retract of βD by $r: \beta D \to R$, then given $f: R \to Z$ and onto $q: Y \to Z$, lift fr to $h: \beta D \to Y$. We have $qh = fr$, whence $qh | R = f$.

THEOREM (GLEASON). *For every object X there exist a projective object P and an onto mapping $f: P \to X$ such that f maps no proper subobject of P onto X. For any other such P' and $f': P' \to X$ there is an equivalence $e: P \to P'$ such that $f = f'e$.*

PROOF. Let F be free and $g: F \to X$ onto. From Lemma 2 we have a minimal subobject P of F such that $g(P) = X$. Let us rename $g | P$ as f; let i denote the inclusion mapping of P into F. Since F is projective, g can be lifted to P, to $q: F \to P$ satisfying $fq = g$. Hence $fqi = gi = f$. If qi were not the identity, Lemma 1 would yield a proper subobject S of P such that $S \cup (qi)^{-1}(S) = P$. Since $qi(qi)^{-1}(S) \subset S$, $f(S)$ contains $fqi(qi)^{-1}(S) = f(qi)^{-1}(S)$, and $f(S) = X$, a contradiction. Thus qi is the identity, P is a retract of F, and P is projective.

Similarly, if P' is projective, $f': P' \to X$ onto, but $f'(S) \neq X$ for all proper subobjects, then we have lifting mappings $e: P \to P'$ and $d: P' \to P$. Since $fde = f$, de is the identity as above; likewise ed is the identity on P', and e is an equivalence.

REFERENCES

1. E. Čech, *On bicompact spaces*, Ann. of Math. vol. 38 (1937) pp. 823–844.

2. A. M. Gleason, *Projective topological spaces*, Illinois J. Math. vol. 2 (1958) pp. 482–489.

3. M. H. Stone, *Algebraic characterization of special Boolean rings*, Fund. Math. vol. 29 (1937) pp. 223–303.

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