A CLASS OF IRREDUCIBLE SYSTEMS OF GENERATORS FOR INFINITE SYMMETRIC GROUPS

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If G is a group and M a subset of G then $\{M\}$ is the smallest subgroup of G containing M. If $\{M\}=G$ then M is a *system of generators for* G. If no proper subset of M is a system of generators for G then M is *irreducible.*

Let N be the set of positive integers; d the cardinal number of N; d^+ the successor of d; $S(d, d^+)$ the group of all one-to-one mappings of N onto itself; $A(d, d)$ the alternating subgroup of $S(d, d^+)$; $S(d, d)$ the finite symmetric subgroup of $S(d, d^+)$.

THEOREM 1. *Let M consist of the sequence of odd length cycles of $S(d, d^+)$*

$$(1, 2, \cdots, n_1), (n_1, n_1+1, \cdots, n_2), \cdots, (n_i, n_i+1, \cdots, n_{i+1}), \cdots$$

with the order of the cycles $s_i=k_i\geqq 3$. Then M is an irreducible system of generators for $A(d, d)$.

PROOF. It is clear from the nature of the set M that $\{M\}\subseteq A(d, d)$. Furthermore, if c_i is removed from M then every element of the group generated by the remaining set leaves the integer $n_{i-1}+1$ fixed. It is sufficient, therefore, to prove that every element of $A(d, d)$ belongs to $\{M\}$. Since the 3-cycles generate $A(d, d)$ we shall show any 3-cycle belongs to M.

Let $x_1<x_2<x_3$ be any triple of elements of N. There exists an element s_i of M such that $x_i\in s_i$ and x_i is not the greatest element of s_i, $i=1, 2, 3$. Furthermore, there exists a positive integer α_i such that $s_i^{\alpha_i}(x_i)=m_i$ where m_i is the largest integer in s_i. In the set M choose the cycle, say s_0, which is the immediate successor of s_3 in the sequence of cycles of M. Denote by $s_{i1}, s_{i2}, \cdots, s_{ir_i}$ the elements of M which occur in the sequence between s_i and s_0. Consider the product

$$s_i^{\alpha_i} s_{i1}^{-1} s_{i2}^{-1} \cdots s_{ir_i}^{-1} s_0 s_{ir_i} \cdots s_{i2} s_{i1} s_i^{t_i-\alpha_i}$$

where t_i is the order of s_i. A computation shows that this product is

$$(x_1, a_2, \cdots, a_p)$$

where $s_0=(a_1, a_2, \cdots, a_p)$. Denote by d_1, d_2, d_3 the three cycles that

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the above formula yields. Now compute $d_1d_2^{-1}$ and $d_1d_3^{-1}$ which yield (x_1, x_2, a_p) and (x_1, x_3, a_p). A final computation of $d_1d_2^{-1}d_3d_1^{-1}$ shows that (x_1, x_2, x_3) belongs to M.

THEOREM 2. *Let M consist of the sequence of cycles of $S(d, d^+)$, where c_1 is of even length,*

$$(1, 2, \cdots, n_1), (n_1, n_1 + 1, \cdots, n_2), \cdots, (n_i, n_i + 1, \cdots, n_{i+1}), \cdots$$

with the order of the cycles $s_i = k_i \geqq 4$. Then M is an irreducible system of generators for $S(d, d)$.

PROOF. By an argument similar to the one given above, it is clear that $A(d, d) \subseteq M$. If x_1, x_2 are any elements of N and $c_1 = (1, 2, \cdots, n_1)$ then $(x_1, x_2)c_1$ is a member of $A(d, d)$, hence in $\{M\}$. But c_1 belongs to M, hence to $\{M\}$ and $(x_1, x_2)c_1c_1^{-1} = (x_1, x_2)$ is in $\{M\}$.

COROLLARY. *There exists d^d irreducible systems of generators for $S(d, d)$ and $A(d, d)$.*

THEOREM 3. *Let M consist of all elements of the form $(i, i+1)$, $i = 1, 2, \cdots, n, \cdots$. Then M is an irreducible system of generators for $S(d, d)$.*

PROOF. Let $r < s$ be any distinct elements of N. Then the formula

$$(r, r + 1)(r + 1, r + 2) \cdots (s - 1, s)(s - 2, s - 1) \cdots (r + 1, r) = (r, s)$$

shows that M contains any transposition. The set M is irreducible because if M_1 is M with $(i, i+1)$ removed then M_1 does not contain $(i+1, x)$ for $x > i+1$.

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