ON A SPECIAL INTEGRAL EQUATION

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1. Introduction. R. H. Cameron posed the following question in a paper [1]. Does

\begin{equation}
(1.1) \quad y(t) = x(t) + \int_0^t [x(s)]^2 ds, \quad 0 \leq t \leq 1,
\end{equation}

have a solution \( x \in C \) for almost every choice of \( y \in C \)? Here \( C \) denotes the space of continuous functions on \( 0 \leq t \leq 1 \) which vanish at \( t = 0 \), and “almost every” means all but a set of Wiener measure zero. The answer is no as we proceed to show.

We will show that if \( y \in N = \{ y \in C : |y(t) + 4t| < 1/10, 0 \leq t \leq 1 \} \) then (1.1) has no solution \( x \) among the elements of \( C \). Then the answer to the question is no, since \( N \), a uniform neighborhood, has positive measure.

Suppose that \( y \in N, x \in C, \) and (1.1) holds. Let

\[ Z(t) = \begin{cases} 
0, & 0 \leq t \leq 1/10 \\
-4(t - 1/10), & 1/10 \leq t \leq \pi/4 + 1/10
\end{cases} \]

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\[ ^2 \text{This question has two other formulations found in [1].} \]

\[ ^3 \text{See, for instance [2].} \]
\[
W(t) = \begin{cases} 
0, & 0 \leq t \leq 1/10 \\
-2 \tan 2(t - 1/10), & 1/10 \leq t < \pi/4 + 1/10
\end{cases}
\]

\[
E = \{t: x(t) \geq W(t), 1/10 \leq t < \pi/4 + 1/10\}
\]

and \(t_1 = \inf E\). It is easy to see that
\[
1/10 < t_1 < \pi/4 + 1/10,
\]

\[
\[x(t)\]^2 \geq [W(t)]^2, \quad 0 \leq t < t_1,
\]

and
\[
x(t_1) = W(t_1).
\]

Since
\[
Z(t) - \frac{3}{10} > y(t), \quad \frac{1}{10} \leq t < \frac{\pi}{4} + \frac{1}{10},
\]

and
\[
Z(t) = W(t) + \int_0^t [W(s)]^2 ds, \quad 0 \leq t < \pi/4 + 1/10
\]
we have using (1.1)
\[
x(t) = y(t) - \int_0^t [x(s)]^2 ds < Z(t) - \frac{3}{10} - \int_0^t [W(s)]^2 ds = W(t) - \frac{3}{10}
\]
for \(1/10 \leq t < t_1\). This is a contradiction since \(x(t_1) = W(t_1)\).

**Bibliography**