A NOTE ON GRADIENT MAPPINGS

E. H. ROTHE[1]

1. **Introduction.** Let E be a real Banach space, and V a (proper or improper) subset of E. Let $D(x, h)$ be a function defined for x in V which for fixed x is a linear continuous functional on E in the variable h, in other words, let $D(x, h)$ define a map of V into the space E^* conjugate to D. Let E and E^* have the topologies induced by their respective norms. The map $D(x, h)$ is called *completely continuous* if it is continuous and if the image of each bounded subset of V has a compact closure in E^*. It is said to be a gradient mapping if there exists a scalar function $I(x)$ such that $D(x, h)$ is the Fréchet differential of $I(x)$ at the point x. For the motivation of this terminology and for details, we refer the reader to [6] or [7]. It is known that there is a close connection between the complete continuity of the gradient on the one hand and the weak continuity or related properties of the scalar $I(x)$ on the other hand [6, Theorems 3.2 and 3.3]. Further literature is quoted in [7, footnote 11]; see also [2] and [3]. In particular, it is known [6, Theorem 3.3] that for convex V the complete continuity of $D(x, h)$ implies the following property of the scalar $I(x)$: to each positive η there corresponds a finite number of elements l_i $(i=1, 2, \cdots, N)$ of E^* such that the inequalities

$$|l_i(h)| < \eta\|h\|/2 \qquad (i = 1, 2, \cdots, N) \tag{1.1}$$

imply

$$|I(x+h) - I(x)| < \eta\|h\|, \qquad h, x+h \text{ in } V. \tag{1.2}$$

For the case where E is a Hilbert space, the converse of this theorem was stated and proved in [6]. The reason for the restriction was as follows: Hilbert spaces are the only Banach spaces (of dimension at least 3) with the property that there exists a projection of norm 1 on every closed linear subspace (see [5]), and such projections were used in the proof given in [6].

However, it will be seen in the present note, that for the purpose at hand it is not necessary to have projections of norm 1 (or of uniformly bounded norm) on *all* closed linear subspaces; rather, it will be sufficient that such projections exist on a large enough collection of subspaces. To be more precise we introduce the following concept.

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DEFINITION 1.1. Let E be a Banach space and E^* the space conjugate to E. We say that E has property P if there exists a set $\{f_i\}$ $(i=1, 2, \cdots)$ of linearly independent elements of E^* and a positive number M with the following two properties: the finite linear combinations of elements of $\{f_i\}$ are dense in E^* (that is, $\{f_i\}$ is a fundamental set in E^* in the sense of Banach [1, p. 58]), and if $N_i = \{x \in E \mid f_i(x) = 0\}$, then for each integer n there exists a projection of norm at most M on the intersection $\bigcap_i^n N_i$.

In §2 we shall show that the converse theorem in question is true under the assumption P; that is we shall prove the following

THEOREM 1.1. *Let E be a Banach space with property P. Let $I(x)$ be a scalar defined in a convex subset V of E with the following two properties: $I(x)$ has a Fréchet differential which is continuous in x, and corresponding to each positive η there exists a finite number of elements $l_1, l_2, \cdots, l_n$ of E^* such that*

$$(1.3) \qquad |I(x+h) - I(x)| < \eta\|h\| \qquad (h, x+h \text{ in } V)$$

for all h for which

$$(1.4) \qquad |l_i(h)| \leqq \eta\|h\| \qquad (i = 1, 2, \cdots, n).$$

Then $D(x, h)$ is completely continuous.

§3 deals with conditions that are sufficient for a Banach space E to have property P. In particular, every reflexive Banach space with a base will be seen to have property P.

2. **Proof of Theorem** 1.1. We first recall some properties of an arbitrary Banach space E. Let $l_1, l_2, \cdots, l_p$ be p linearly independent elements of E^*, and let $N = \{x \in E \mid l_i(x) = 0, i = 1, 2, \cdots, p\}$. Moreover let $a_1, a_2, \cdots, a_p$ be any set of p elements of E which are linearly independent mod N. (This means that $\sum_{i=1}^n a_i\gamma_i \in N$ implies that the real numbers γ_i are all 0.) Finally let E^p be the space spanned by the a_i. Then each x in E admits a unique decomposition

$$(2.1) \qquad x = x_1 + n \qquad (x_1 \in E^p, n \in N),$$

and, on account of the linear independence of the l_i, the determinant of the $l_i(a_j)$ is different from zero. It is easily seen that this latter fact implies the existence of a base $b_1, b_2, \cdots, b_p$ of E^p which is "orthogonal," that is, for which

$$(2.2) \qquad l_i(b_j) = \delta_{ij} \qquad (i, j = 1, 2, \cdots, p),$$

where δ_{ij} is the Kronecker symbol. If then $l(x)$ is an arbitrary linear

continuous functional on E, one obtains almost immediately the representation

$$l(x) = \sum_{j=1}^{p} l_j(x) l(b_j) + l(n), \tag{2.3}$$

(apply l to (2.1) after x_1 has been expressed in terms of the b_j, and use (2.2)). With the notation of this paragraph, the following lemma is an immediate consequence of the representation (2.3); it may be considered as a generalization of the well-known fact that any l vanishing on N is a linear combination of the l_i.

LEMMA 2.1. *Let $l(x)$ have the property that there exists a positive number η such that*

$$|l(y)| < \eta \|y\| \qquad \textit{for all } y \textit{ in } N. \tag{2.4}$$

Then, for all x in E,

$$l(x) = \sum_{i=1}^{p} \alpha_i l_i(x) + R(x) \tag{2.5}$$

with suitable constants α_i and with

$$|R(x)| \leqq \eta \|n\|, \tag{2.6}$$

where n is the element of N appearing in the decomposition (2.1) *of x.*

In addition to Lemma 2.1 we shall need the following

LEMMA 2.2. *If the assumption in Lemma* 2.1 *holds, and if there exists a projection π of E on N of norm at most M, then the representation* (2.5) *holds with*

$$|R(x)| \leqq M\eta \|x\|. \tag{2.7}$$

PROOF. Since the elements b_i are linearly independent mod N, the elements $b_i' = b_i - \pi b_i$ have the same property. If then $'E^p$ is the space spanned by the b_i', we obtain instead of (2.1) the decomposition

$$x = x_1' + n' \qquad (x_1' \in {}'E^p,\ n' \in N). \tag{2.8}$$

It is easily verified that $x_1' = x - \pi(x)$ and $n' = \pi(x)$, and consequently, by assumption,

$$\|n'\| \leqq M\|x\|. \tag{2.9}$$

If we now apply Lemma 2.1 to the new decomposition (2.8), we have to replace n by n' in (2.6). The inequality thus obtained together with (2.9) proves (2.7).

We now turn to the proof of Theorem 1.1. We have to establish the complete continuity of the gradient mapping $D(x, h)$. For this purpose we may without loss of generality assume that V is bounded. Therefore, by [6, Lemma 3.2] it will be sufficient to prove that, corresponding to a given $\epsilon > 0$, there exists a mapping $D'(x, h)$ of V into E^* with the following two properties: the image of V under D' is contained in a finite-dimensional subspace of E^*, and

$$| D(x, h) - D'(x, h) | \leqq \epsilon \|h\|. \tag{2.10}$$

We first choose, corresponding to the given ϵ,

$$\eta = \epsilon / M, \tag{2.11}$$

where M is the number appearing in Definition 1.1. Corresponding to η, there exist by assumption elements l_i of E^* such that (1.4) implies (1.3). Now, by Definition 1.1, the set $\{f_i\}$ is fundamental in E^*. Consequently there exist finite linear combinations f_i' of elements of the set $\{f_i\}$ such that $\|l_i - f_i'\| < \eta$, in other words, such that

$$| l_i(h) - f_i'(h) | < \eta \|h\| \qquad (i = 1, 2, \cdots, n). \tag{2.12}$$

Now let the integer p be such that the set $f_1, f_2, \cdots, f_p$ contains all elements of $\{f_i\}$ which occur in the linear combinations f_i' $(i = 1, 2, \cdots, n)$, and let $N = \{h | f_i(h) = 0, i = 1, 2, \cdots, p\}$. Then, for h in N, all $f_i'(h)$ are zero, and (2.12) shows that (1.4) holds for such h; by assumption this implies (1.3). But from (1.3) together with the definition of the Fréchet differential, one concludes easily (see [6, p. 430]) that

$$| D(x, h) | \leqq \eta \|h\| \tag{2.13}$$

for all h in N. We now may apply Lemma 2.2, since by Definition 1.1 there exists a projection of E on N of norm at most M. In the present notation, we thus obtain from (2.5), (2.7), and (2.11)

$$D(x, h) = \sum_{i=1}^{p} \alpha_i f_i(h) + R(h), \qquad | R(h) | \leqq M\eta \|h\| \leqq \epsilon \|h\|.$$

This shows that, with the definition

$$D'(x, h) = \sum_{i=1}^{p} \alpha_i f_i(h),$$

$D'(x, h)$ satisfies the two requirements formulated at the beginning of this proof.

3. **Sufficient conditions for property** P. Let E be a Banach space with a base $\{b_i\}$. Then there exists a unique sequence $\{f_i\}$ of elements of E^* such that, for every element x of E,

$$x = \sum_{i=1}^{\infty} b_i f_i(x) \tag{3.1}$$

[**1**, Chapter VII, §3]. We now make the assumption that the set $\{f_i\}$ is fundamental in E^*, and we claim that then E has property P. Indeed, if $N^p = \{x \in E \mid f_i(x) = 0 \text{ for } i = 1, 2, \cdots, p\}$, then N^p is the space spanned by $b_{p+1}, b_{p+2}, \cdots$. It follows from Banach's results [**1**, p. 111] that the map $E \to N^p$ mapping the point (3.1) of E into the point

$$\sum_{i=p+1}^{\infty} b_i f_i(x)$$

is a projection with a norm admitting a bound independent of p. This shows that E has property P.

It follows in particular that every reflexive Banach space with a base has property P. For in such spaces, our assumption that the set $\{f_i\}$ is fundamental is satisfied; this follows from [**4**, Lemma 1, p. 70 together with Theorem 3, p. 71].

For reflexive Banach spaces with a base, Citlanadze [**2**] stated without proof some propositions related to Theorem 1.1. In a later paper [**3**, Theorem 1] he proved such a theorem in the more special case of an L_p space ($p > 1$).

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The University of Michigan