SOLUTIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS
WHICH ARE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER

LAWRENCE GOLDMAN

Let $\mathcal{F}$ be an ordinary differential field (i.e., a field with a given derivation) of characteristic zero. An element $z$ belonging to a differential field extension of $\mathcal{F}$ is said to be of order $r$ over $\mathcal{F}$ if the lowest order irreducible differential equation with coefficients in $\mathcal{F}$ that $z$ satisfies is of order $r$. It follows that $z$ is of order $r$ over $\mathcal{F}$ if and only if the degree of transcendency of $\mathcal{F}(z)$ over $\mathcal{F}$ is $r$.

In [4] Ritt proved that if $P, Q \in \mathcal{F}\{y\}$ and $Q$ vanishes for every zero of $P$ then the sum of the lowest (highest) degree terms of $Q$ vanishes for every zero of the sum of the lowest (highest) degree terms of $P$.

For our purpose we need the following slight generalization; it can be obtained either by essentially the same proof as used by Ritt, or as an almost immediate corollary of this theorem.

**Theorem 1.** Let $P, Q \in \mathcal{F}\{y\}$ and let $Q$ vanish for every zero of $P$ which is of order $\geq r$ over $\mathcal{F}$; then the sum of lowest (highest) degree terms of $Q$ vanishes for every zero of the sum of lowest (highest) degree terms of $P$ which is of order $\geq r$ over $\mathcal{F}$.

**Theorem 2.** Let $z$ be a zero of an $n$th order linear differential polynomial $L(y) \in \mathcal{F}\{y\}$ and let the order of $z$ over $\mathcal{F}$ be 1. Then there exists an integer $r$, $0 \leq r < n$, such that $z(r)$ is a zero of an irreducible first order differential polynomial $P(y) \in \mathcal{F}\{y\}$ the sum of whose highest degree terms is of order 1 and $z(r)$ is of order 1 over $\mathcal{F}$.

**Proof.** Since $z$ is of order 1 over $\mathcal{F}$, $\mathcal{F}(z) = \mathcal{F}(z, z')$, which is an algebraic function field of one variable over $\mathcal{F}$. Let $v$ be an infinite valuation on $\mathcal{F}(z, z')$ such that $v$ is trivial on $\mathcal{F}$ and for any $Q(z)$, $R(z) \in \mathcal{F}[z]v(Q/R) = \text{degree of } R - \text{degree of } Q$. Since $z$ is a zero of an $n$th order linear differential polynomial we can not have $v(z^{(s+1)}) < v(z^{(s)})$ for all $s$ less than $n$. Let $r$ be the smallest integer such that $v(z^{(s+1)}) \geq v(z^{(r)})$; then the order of $z^{(r)}$ over $\mathcal{F}$ is 1. For, $z^{(r)} \in \mathcal{F}(z)$ so that the order of $z^{(r)}$ over $\mathcal{F}$ is $\leq 1$; if $z^{(r)}$ were algebraic over $\mathcal{F}$ then $v(z^{(r)})$ would be zero, which is greater than $v(z)$, contradicting our assumption on the minimality of $r$. Let $P(y) \in \mathcal{F}\{y\}$ be the first order ir-
reducible differential polynomial which vanishes for \( z^{(r)} \). Since at least two terms of \( P(z^{(r)}) \) must have the same smallest value under the given valuation \( \nu \), it follows that the highest degree terms of \( P(y) \) are of order 1.

A homogeneous linear differential polynomial \( L(y) \in \mathbb{F}\{y\} \) is said to be linearly reducible over \( \mathbb{F} \) if there exist homogeneous linear differential polynomials \( M(y), N(y) \in \mathbb{F}\{y\} \), each one of positive order, such that

\[
L(y) = M(N(y)).
\]

If no such decomposition exists we say \( L(y) \) is linearly irreducible over \( \mathbb{F} \).

**Theorem 3.** Let \( L(y) \in \mathbb{F}\{y\} \) be a homogeneous linear differential polynomial linearly irreducible over \( \mathbb{F} \). If a zero \( z \) of \( L(y) \) is of order 1 over \( \mathbb{F} \), then there exists a fundamental system of zeros \( (u_1, \cdots, u_n) \) of \( L(y) \) such that \( u_i/u_i, i=1, \cdots, n \), is algebraic over \( \mathbb{F} \).

**Proof.** Since \( L(y) \) is linearly irreducible over \( \mathbb{F} \) it suffices to show the existence of one zero \( u \) of \( L(y) \) such that \( u'/u \) is algebraic over \( \mathbb{F} \) [2]. By Theorem 2 there exists an integer \( r \) such that \( z^{(r)} \) is a zero of a first order differential polynomial \( P(y) \in \mathbb{F}\{y\} \) the sum of whose highest degree terms is of order 1. Because \( L(y) \) is linearly irreducible there exist homogeneous linear differential polynomials \( M(y), N(y) \in \mathbb{F}\{y\} \), \( M(y) \) linearly irreducible over \( \mathbb{F} \), such that \( M(z^{(r)}) = 0 \) and, for any nontrivial zero \( w \) of \( M(y) \), \( N(w) \) is a nontrivial zero of \( L(y) \) [5, vol. 2, pp. 164, 165]. By Theorem 1 there exists a nontrivial zero \( w \) of the sum of the highest degree terms of \( P(y) \) such \( M(w) = 0 \). Since \( w \) is a zero of a homogeneous first order differential polynomial \( w'/w \) is algebraic over \( \mathbb{F} \). Let \( u = N(w) \) then \( L(u) = 0 \). Since \( w'/w \) is algebraic over \( \mathbb{F} \), \( u = kw \), where \( k \) belongs to an algebraic extension of \( \mathbb{F} \), so that \( u'/u \) is algebraic over \( \mathbb{F} \) and our theorem follows.

**Definition.** If the lowest order linear differential polynomial which vanishes for \( z \) is of order \( n \), then we say that the linear order of \( z \) over \( \mathbb{F} \) is \( n \). Let the linear order of \( z \) over \( \mathbb{F} \) be \( n \) and let \( V \) be the set of all linear differential polynomials in \( z \) with coefficients in \( \mathbb{F} \). \( V \) is, in an obvious way, an \( n+1 \) dimensional vector space over \( \mathbb{F} \) so that the linear order of any element of \( V \) over \( \mathbb{F} \) is \( \leq n \).

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2 Loewy assumes that \( \mathbb{F} \) has an algebraically closed field of constants and uses in his proof the automorphisms of a Picard-Vessiot extension of \( \mathbb{F} \). It is easily seen, however, that by the substitution of relative isomorphisms over \( \mathbb{F} \) for his automorphisms, the proof remains valid. For, we can use the theorem by Kolchin [1] that an element \( t \) belonging to \( \mathbb{K} \), a differential field extension of \( \mathbb{F} \), which is left invariant by every isomorphism of \( \mathbb{K} \) over \( \mathbb{F} \), belongs to \( \mathbb{F} \).
Theorem 4. Let \( z \) be a zero of a first order differential polynomial \( P(y) \in \mathbb{F}\{y\} \) and of a linear differential polynomial \( L(y) \in \mathbb{F}\{y\} \), and let \( \mathbb{F} \) be the algebraic closure of \( \mathbb{F} \). There exists \( u \in \mathbb{F}(z) \) such that \( z \) is algebraic over \( \mathbb{F}(u) \) and the linear order of \( u \) over \( \mathbb{F} \) is 1.

Remark. If a first order differential polynomial \( P(y) \in \mathbb{F}\{y\} \) factors over \( \mathbb{F} \) into linear factors, it is well known that any zero \( u \) of \( P(y) \) is a zero of a linear differential polynomial of higher order with coefficients in \( \mathbb{F} \) [3]. Also, any polynomial \( z \in \mathbb{F}\{u\} \) is a zero of a first order differential polynomial \( Q(y) \in \mathbb{F}\{y\} \) and of a linear differential polynomial with coefficients in \( \mathbb{F} \). \( Q(y) \) may remain irreducible over \( \mathbb{F} \). For example, let \( \mathbb{F} \) be the field of rational numbers, \( u = e^z \), \( z = e^z + e^{2z} \); \( Q(y) = (2y - y')^2 - (y' - y) \) is, obviously, irreducible over \( \mathbb{F} \).

Our theorem states that besides the obvious cases just mentioned there is only one more possibility for an element \( z \) to be simultaneously a zero of a first order and of a linear differential polynomial; namely that \( z \) belongs to an algebraic extension of \( \mathbb{F}(u) \) where \( u \) is a zero of a first order linear differential polynomial with coefficients in \( \mathbb{F} \).

Proof of Theorem. If \( z \in \mathbb{F} \) we take \( u = z \). Let the order of \( z \) over \( \mathbb{F} \) (and hence over \( \mathbb{F} \)) be 1. Let \( V \) be the vector space of all linear differential polynomials in \( z \) with coefficients in \( \mathbb{F} \). Since \( z \) is a zero of a linear differential polynomial, \( V \) is a finite dimensional vector space. For any element \( v \in V \), \( \mathbb{F}(v) \subseteq \mathbb{F}(z) \) so that order of \( v \) over \( \mathbb{F} \) is \( \leq 1 \). Let \( A \) be the set of all elements \( v \) in \( V \) such that order of \( v \) over \( \mathbb{F} \) is 1. \( A \) is not empty since \( z \in A \). Of all the elements in \( A \) choose \( u \) such that the linear order of \( u \) over \( \mathbb{F} \) is least. We are going to show that the linear order of \( u \) over \( \mathbb{F} \) is 1.

Let the linear order of \( u \) over \( \mathbb{F} \) be \( n \) and let \( W \) be the \( n+1 \) dimensional vector space over \( \mathbb{F} \) of all linear differential polynomials in \( u \) with coefficients in \( \mathbb{F} \). For any \( w \in W - \mathbb{F} \) the following holds:

1. \( w \in A \) (i.e. \( w \) is of order 1 over \( \mathbb{F} \)).
2. Linear order of \( w \) over \( \mathbb{F} \) is \( n \).
3. If the \( n \)th order linear equation that \( w \) satisfies over \( \mathbb{F} \) is \( M(y) = f \), \( M(y) \) homogeneous, \( f \in \mathbb{F} \); then \( M(y) \) is linearly irreducible.

To prove (1) note that \( w \in V - \mathbb{F} \) and, since \( \mathbb{F} \) is algebraically closed the order of \( w \) over \( \mathbb{F} \) is 1. Since \( w \in A \) the linear order of \( w \) over \( \mathbb{F} \) is \( \geq n \) (since \( n \) was least) but \( w \in W \) and each element in \( W \) has linear order over \( \mathbb{F} \leq n \). This proves (2). To prove (3) we note that if \( M(y) = N_2(N_1(y)) \), \( N_1(y) \) of positive order, then the linear order of \( N_1(w) \)
over $\bar{\mathbb{F}}$ is the order of $N_2$ which is less than linear order of $w$ over $\bar{\mathbb{F}}$ contradicting (2). Hence $M(y)$ is linearly irreducible over $\bar{\mathbb{F}}$.

Now, by Theorem 2, there exists $w\in W-\bar{\mathbb{F}}$ such that $w$ is a zero of a first order differential polynomial $Q(y)\in \bar{\mathbb{F}}\{y\}$ the sum of whose highest degree terms is of order 1. Since $\bar{\mathbb{F}}$ is algebraically closed the sum of the highest degree terms factors into linear factors with at least one of the factors $N_1(y)$ of order 1. By Theorem 1 a generic zero of the prime differential ideal generated by $N_1(y)$ is a zero of $M(y)$ (since the generic zero is of order 1 over $\bar{\mathbb{F}}$), so that $M(y)$ belongs to the prime differential ideal $\{N_1(y)\}$. Since $M(y)$ is linear (i.e. of the same degree as $N(y)$), $M(y) = N_2(N_1(y))$. By (3) $M(y)$ is linearly irreducible so that $N_2(y)$ is of order zero and $M(y)$ is of order 1. By (2) this implies that $n=1$ and the linear order of $u$ over $\bar{\mathbb{F}}$ is 1. Since $u$ is of order 1 over $\bar{\mathbb{F}}$ it follows that $z$ is algebraic over $\bar{\mathbb{F}}\langle u \rangle$; this proves our theorem.

**Corollary.** If a zero $z$ of a first order differential polynomial $P(y)\in \mathbb{F}\{y\}$ is a zero of a linear differential polynomial $L(y)\in \mathbb{F}\{y\}$, then either $z$ is algebraic over $\mathbb{F}$ or $P(y)$ is solvable by quadratures.

**References**


**Stevens Institute of Technology**