ON A PROBLEM OF WÖLK IN INTERVAL TOPOLOGIES

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1. Introduction. Let $P$ be a partially ordered set (poset) with respect to a relation $\leq$. We say that two elements $x$ and $y$ in $P$ are incomparable if and only if $x \nleq y$ and $x \ngeq y$. Let us call a subset $M$ of $P$ diverse if and only if $x \in M$, $y \in M$, and $x \neq y$ imply that $x$ and $y$ are incomparable. We define the width of $P$ to be $\text{l.u.b.} \{k \mid k$ is the cardinal number of a diverse subset of $P\}$.

We shall call a subset $M$ of $P$ Dedekind-closed if and only if whenever $D$ is an up-directed subset of $M$ and $y = \text{l.u.b.} D$ or $D$ is a down-directed subset of $M$ and $y = \text{g.l.b.} D$, we have $y \in M$. We define a topology $\mathcal{S}$ on $P$ whose closed sets are precisely the Dedekind-closed subset of $P$ and let $\mathcal{S}$ denote the interval topology on $P$, which is obtained by taking all sets of the form $[a, b]$ as a sub-basis for the closed sets.

E. S. Wölk introduced the following concept [1]:

**Definition.** If $\mathfrak{T}$ is a topology defined on $P$, we shall say that $\mathfrak{T}$ is order-compatible with $P$ if and only if

(i) every set closed with respect to $\mathfrak{T}$ is Dedekind-closed, and
(ii) every set of the form $\{x \in P \mid a \leq x \leq b\}$ is closed with respect to $\mathfrak{T}$.

He proved the following theorem in his paper [1].

**Theorem.** If $P$ is a poset of finite width, then $P$ possesses a unique order-compatible topology.

And he proposed the question: "Whether, in the above theorem, the hypothesis that $P$ is of finite width, can be replaced by the weaker condition that $P$ contains no infinite diverse subset."

The main purpose of this note is to give the answer to the above question, and it is contained in the following theorems.

**Theorem 1.** If $P$ contains no infinite diverse set, then $P$ possesses a unique order-compatible topology.

**Theorem 2.** Let $P$ be a complete lattice. Then $P$ possesses a unique order-compatible topology if and only if $P$ contains no infinite diverse set.

2. Main theorems. First we shall prove the following lemma which is the main result in this paper.

Received by the editors February 18, 1959.
Lemma. Let $P$ be a poset containing no infinite diverse set and $f$ be a net on $A$ with range $(f) = S \subset P$. If element $y$ of $P$ is the l.u.b. of the range of every subnet of $f$, then there exists an up-directed set $M \subset S$ such that $y = \text{l.u.b.} (M)$.

Proof. Let us suppose that the lemma is false. Let $M_1$ be any maximal up-directed subset of $S$ (which exists by Zorn's lemma). By the assumption that the lemma is false, we have $y \neq \text{l.u.b.} M_1$. Hence there exists no subnet of $f$ with range contained in $M_1$. Therefore there exists $\alpha_1 \in A$ such that $f(\alpha) \in S - M_1$ for all $\alpha \geq \alpha_1$. Next let us choose a maximal up-directed subset $M_2$ of $\{f(\alpha) | \alpha \geq \alpha_1\}$. Then we have $y \neq \text{l.u.b.} M_2$ and there exists $\alpha_2 \in A$ such that $f(\alpha) \in S - M_1 - M_2$ for all $\alpha \geq \alpha_2 \geq \alpha_1$. Now choose $M_3$, a maximal up-directed subset of $\{f(\alpha) | \alpha \geq \alpha_2\}$, and continue the above process.

Thus we obtain a countable infinite set of maximal up-directed subsets: $M_1, M_2, M_3, \ldots$. From the fact that $M_i$ is a maximal up-directed set, we have

$$(*) \ x \in M_i, \ y \in M_j \ \text{imply} \ x \nleq y. \text{ More generally, } x \in M_i, \ y \in M_j, \ i < j \ \text{imply} \ x \nleq y.$$

If for all pairs $x \in U_i, \ M_i; \ y \in U_i, \ M_i$, there is an element $z$ of $U_i, \ M_i$ such that $x \leq z, \ y \leq z$, then $U_i, \ M_i$ is an up-directed set which contradicts the fact that each $M_i$ is a maximal up-directed set. Therefore, there exist elements $a_i \in M_i$ and $b_i \in M_i$ such that $U_i, \ M_i$ contains no $z$ such as $z \geq a_i$ and $z \geq b_i$. By the definition of $a_i$ and $b_i$ there exists an infinite number of $M_i$ which contains no upper bound of $a_i$ or an infinite number of $M_i$ which contains no upper bound of $b_i$. In fact, if there only exists a finite number of $M_i$ which contains no upper bound of $a_i$ and $M_j$ which contains upper bound of $b_i$, then there exists $M_s$ containing an upper bound of $a_i$ and $b_i$ which contradicts the definition of $a_i$ and $b_i$. For example, if there exists an infinite number of $M_i$ (max, $(k, 1) \leq i$) which contains no upper bound of $b_i$, then we put $c_i = b_i$ and denote such $M_i$ by $M^1, M^2, M^3, \ldots$ (in the same order as $M_i$).

Similarly, there exist elements $a_2 \in M^2_i$ and $b_2 \in M^2_i$ such that $U_i, M^2_i$ does not contain $z$ such as $z \geq a_2, \ z \geq b_2$. By the definition of $a_2$ and $b_2$, there exists either an infinite number of $M^2_i$ which contains no upper bound of $a_2$ or an infinite number of $M^2_i$ which contains no upper bound of $b_2$. For example, if there exists an infinite number of $M^2_i$ (max $(k', l') \leq i$) which contains no upper bound of $a_2$, then we put $c_2 = a_2$ and denote such $M^2_i$ by $M^3, M^2, M^3, \ldots$ (in the same order as $M_i$).

Continuing this process, we have an infinite set $c_1, c_2, \ldots$. Set
\{c_i\} is an infinite diverse set of P. In fact, by the definition of \(M_t\), \(x \in M_t^n, y \in M_t^k\) and \(i < k\) imply \(x \preceq y\). Hence by the definition of \(c_k\) we have \(c_i \preceq c_k\) for \(i < k\). Since each of \(M_t^n\) contains no upper bound of \(c_{n-1}\), we have \(c_i \preceq c_k\) for \(i < k\). Therefore we have \(c_i \preceq c_k\) for \(i \neq k\). The proof is complete.

We obtain from the above lemma the following theorem.

**Theorem 1.** If P contains no infinite diverse set, then P possesses a unique order-compatible topology.

The proof will not be given since it is exactly the same as for the proof of Wolk's Theorem 1 in [1].

**Theorem 2.** Let P be a complete lattice. Then, P possesses a unique order-compatible topology if and only if P contains no infinite diverse set.

**Proof.** Since P is a complete lattice, P is compact in the interval topology [2]. Now, suppose that P possesses a unique order-compatible topology, then P is compact in the \(\mathcal{D}\)-topology. Suppose that \(\{a_i|i=1, 2, \ldots\}\) is an infinite diverse subset of P. Let \(F_n = \{a_i|i \geq n\}\). Then \(F_n\) is closed in the \(\mathcal{D}\)-topology and the family of all \(F_n\) has the finite intersection property. But \(\bigcap F_n\) is empty which is a contradiction.

Since the necessity of the condition is Theorem 1, then the theorem is proved.

**References**


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