UPPER AND LOWER COMPLEMENTATION IN A MODULAR LATTICE

S. P. AVANN

In this paper we define upper and lower complements of $a \in L$, always a finite modular lattice, such that these become an ordinary complement of a when L is complemented (Theorem 8). Uniqueness of upper or of lower complements for all $a \in L$ implies L is distributive (Theorem 5). Our principal result is a set of 8 equivalent conditions for uniqueness of upper or lower complement of a particular element $a \in L$ (Theorem 4).

We shall employ $\supset$, $\supseteq$, $>$ for proper inclusion, inclusion, and covering respectively, and $\subset$, $\subseteq$, $<$ for their duals. Unit and zero of L will be u and z respectively. Otherwise, notation and terminology of Birkhoff's *Lattice theory* [2] will be adhered to.

We shall denote by A the set $A = \{x \in P \mid x \subseteq a\}$ where P is the set of join-irreducibles of L partially ordered by the ordering relation of L. Elementary properties enjoyed by these sets A are: (1) $a = \bigcup_{x \in A} x$; (2) $A = B$ if and only if $a = b$; (3) $A \supset B$ if and only if $a \supset b$; (4) $c = a \cup b$ and $d = a \cap b$ imply $C \supseteq A + B$ and $D = A \cdot B$ where $(+)$ and $(\cdot)$ are point-set sum and product.

We denote by a^* the join of all elements covering a and by a_* the meet of all elements covered by a. We define $u^* = u$ and $z_* = z$.

The quotient c/a is an upper transpose of b/d and b/d is a lower transpose of c/a if and only if $c = a \cup b$ and $d = a \cap b$.

In Lemma 1, Theorems 5 and 8, and Corollary to Theorem 9 use of parenthesized words yields the dual theorem.

LEMMA 1. *In a modular lattice, c/a (b/d) is a maximal (minimal) complemented quotient in the complete set Q of projective quotients to which it belongs if and only if $c = a^*$ $(d = b_*)$.*

PROOF. First suppose c/a is a maximal complemented quotient and consider $a_i > a$. Then $a_i \subseteq c$; otherwise $a_i \cap c = a$ and $c \cup a_i / a_i$ is a proper upper transpose of c/a contradicting the maximality of c/a. Applying Theorem 6 [2, p. 105] we obtain $c = \bigcup_{a_i > a} a_i = a^*$. Conversely, if $c = a^*$, then c/a is complemented by the same Theorem 6. Assume there exists a proper upper transpose e/f of c/a. Then for some a_1, $f \supseteq a_1 > a = f \cap c$. But $a_1 \subseteq a^* = c$ by definition, which leads to the contradiction $a_1 \subseteq f \cap c = a$. Hence c/a has no proper upper transpose.

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DEFINITION 1. In a modular lattice the numerator a'_U of a minimal quotient projective with a^*/a is called an upper complement of a. It is called a direct upper complement if the minimal quotient is a lower transpose of a^*/a, otherwise a'_U is called indirect. Dually, the denominator a'_L of a maximal quotient projective with a/a_*, is called a lower complement of a. It is called direct if the maximal quotient is an upper transpose of a/a_*, otherwise indirect. An upper associate a_U of a is the denominator of a maximal quotient projective with a^*/a, and a lower associate a_L of a is the numerator of a minimal quotient projective with a/a_*.

In view of Lemma 1 a is both an upper and a lower associate of itself.

THEOREM 1. *In a modular lattice b is a (direct) upper complement a'_U of a if and only if a is a (direct) lower complement b'_L of b.*

PROOF. Suppose $b=a'_U$ is determined by the minimal quotient b/d projective to a^*/a, the latter being complemented by Lemma 1. Projective quotients are isomorphic, so that b/d is also complemented as well as minimal. Hence $d=b_*$ by Lemma 1. Again by Lemma 1 a^*/a is maximal and is projective to b/b_*. Hence $a=b'_L$. The converse follows by duality. The relationship between a and b is direct if and only if a^*/a and b/b_* are upper and lower transposes of one another respectively.

THEOREM 2. *In a modular lattice L there exists at least one direct upper complement and one direct lower complement of each $a\in L$.*

PROOF. The theorem follows by the transitivity of lower and upper transposition. If a^*/a has no proper lower transpose, a^* is the unique upper complement of a, and dually.

THEOREM 3. *In a modular lattice L the number of upper (lower) complements of each $a\in L$ is equal to the number of upper (lower) associates of a.*

PROOF. The theorem follows directly from Definition 1 and Theorem 6.2 of [1], which asserts that in a complete set Q of projective complemented quotients, the number of maximal quotients is equal to the number of minimal quotients.

COROLLARY. *For $b=a'_U$ and $a=b'_L$ in a modular lattice, the number of upper complements of a is equal to the number of lower complements of b.*

THEOREM 4. *In a modular lattice L let b and a be respectively a direct*

upper complement and a direct lower complement of the other and let Q be the complete set of projective complemented quotients to which a^*/a and b/b_* belong. The following conditions are equivalent.

(A) *b is the unique upper complement of a.*

(B) *a is the unique lower complement of b.*

(C) *Every $e/f \in Q$ is a lower transpose of a^*/a and an upper transpose of b/b_*.*

(D) *a^*/b_* is (isomorphic to) the direct product $a/b_* \times b/b_*$.*

(E) *a and b are a distributive pair: all 6 distributive laws hold for a, b and every $g \in L$.*

(F) $A^* = A + B$.

(G) $A^* - A = B - B_*$.

(H) *$E - F$ is an invariant subset of P, the set of all join-irreducible elements of L, for all $e/f \in Q$.*

PROOF. (A) implies (B). Referring to Theorem 1 and its proof, when b is the only upper complement of a, b/b_* is the unique minimal quotient of Q. From [**1**, Theorem 6.2] quoted in the proof of Theorem 3 we conclude that there exists exactly one maximal quotient of Q, which must be a^*/a by Lemma 1. Hence a is the unique lower complement of b.

(B) implies (A) by duality.

(A) and (B) imply (C). Suppose there exists in Q a quotient e/f which is not a lower transpose of a^*/a. By Lemma 1 a maximal upper transpose of e/f is of the form g^*/g and is maximal in Q. Since lower transposition is transitive, g^*/g must also fail to be a lower transpose of a^*/a. Thus g is a lower complement of b and is distinct from a, contradicting hypothesis (B). Thus every $e/f \in Q$ is a lower transpose of a^*/a and by a dual argument is an upper transpose of b/b_*.

(C) implies (A) and (B) since a^*/a and b/b_* are obviously the unique maximal and the unique minimal quotients respectively of Q.

(C) implies (D). As an immediate corollary of Theorem 7 [**2**, p. 73] and its dual, (D) is valid if and only if for every $k \in a^*/b_*$ $k = (k \cap a) \cup (k \cap b) = (k \cup a) \cap (k \cup b)$. Assume (D) is false and that h is minimal in a^*/b_* such that $h \supset (h \cap a) \cup (h \cap b)$. Obviously $h \supset b_*$. Let g be an arbitrary element of a^*/b_* covered by h. Then $h \supset (h \cap a) \cup (h \cap b) \supseteq (g \cap a) \cup (g \cap b) = g$. It follows that $g = (h \cap a) \cup (h \cap b)$ and that g is unique; i.e. h is a join-irreducible of a^*/b_* (h covers only one element g). Moreover, in a^*/b_* g, $b \supseteq h \cap b \supseteq g \cap b$ from which $h \cap b = g \cap b$. Next, $h > g = g \cup (b \cap g) = g \cup (b \cap h) = (g \cup b) \cap h$ by the modular axiom. Therefore by the upper semimodularity axiom $h \cup b = (g \cup b) \cup h > g \cup b$. We observe for later reference that $h \cup b/h$ is an upper transpose of $g \cup b/g$. Now $t \leftrightarrow t \cap a$ deter-

mines Dedekind's natural isomorphism [2, Theorem 6, p. 73] between a^*/b and its lower transpose a/b_*. Hence $(h\cup b)\cap a>(g\cup b)\cap a$ in a/b_*. Validity of one distributive law $g\cap(a\cup b)=g\cap a^*=g=(g\cap a)\cup(g\cap b)$ implies validity of all 6. Hence $(h\cup b)\cap a>(g\cup b)\cap a=(g\cap a)\cup(b\cap a)=(g\cap a)\cup b_*=g\cap a$. Again, $(g\cap a)\cup b=(g\cup b)\cap(a\cup b)=(g\cup b)\cap a^*=g\cup b$, together with $(g\cap a)\cap b=g\cap b_*=b_*$, show that $g\cup b/g\cap a$ is an upper transpose of $b/b_*\in Q$ and therefore also complemented. Hence by [2, p. 105, Theorem 6, L7′] $g\cup b$ is the join of elements covering $g\cap a$. Applying the modular axiom, $h\cup b=a^*\cap(h\cup b)=[(g\cup b)\cup a]\cap(h\cup b)=(g\cup b)\cup[a\cap(h\cup b)]$ likewise is a join of elements covering $g\cap a$. Thus $h\cup b/g\cap a$ satisfies L7′ and is complemented. Since h is a join-irreducible of a^*/b_*, it is a join-irreducible of the complemented quotient sublattice $h\cup b/g\cap a$ and must cover $g\cap a$. Thus $h>g=g\cap a$. We have now shown $h\cup b/h$ is an upper transpose of $g\cup b/g=g\cup b/g\cap a$, which in turn is an upper transpose of b/b_*. Hence by transitivity of upper transposition and hypothesis (C) a^*/a is an upper transpose of $h\cup b/h$ so that $h\subseteq a$. Thus the initial assumption that (D) is false leads to the contradiction $g=(h\cap a)\cup(h\cap b)=h\cup(h\cap b)=h$.

(D) implies (C). Let e/f be arbitrary in Q. By the extended semi-modularity axiom [2, p. 100, (2)] and its dual there exists between a^*/a and e/f a sequence of upper and lower transposes, each a "covering" transpose: $a^*/a=e_0/f_0\sim e_1/f_1\sim\cdots\sim e_n/f_n=e/f$ where $e_{i-1}>e_i$ and $f_{i-1}>f_i$ or $e_{i-1}<e_i$ and $f_{i-1}<f_i$ $(i=1, 2, \cdots, n)$. We shall prove by an induction on i that all these quotients satisfy (C). Trivially for $i=0$, (C) is satisfied. Assume (C) is satisfied for i. Suppose e_{i+1}/f_{i+1} is an upper covering transpose of e_i/f_i, hence is also an upper transpose of b/b_*. Assume $a\not\supseteq f_{i+1}$. Then $f_{i+1}\supset a\cap f_{i+1}\supseteq f_i$ demands $f_{i+1}>a\cap f_{i+1}=f_i$. By the upper semi-modularity axiom $a\cup f_{i+1}>a$. But then $a\cup f_{i+1}\subseteq a_1^*$. Then $f_{i+1}=a^*/b_*=a/b_*\times b/b_*$ requires $f_{i+1}=(a\cup f_{i+1})\cap(b\cup f_{i+1})=(a\cup f_{i+1})\cap e_{i+1}$. But $(a\cup f_{i+1})\cup e_{i+1}=(a\cup f_{i+1})\cup(b\cup f_{i+1})=a^*\cup f_{i+1}=a^*$. We have verified that $a^*/a\cup f_{i+1}$ is an upper transpose of e_{i+1}/f_{i+1}. Hence a^*/a is projective with and therefore isomorphic to a proper sublattice $a^*/a\cup f_{i+1}$ of itself, a contradiction. Thus $a\supseteq f_{i+1}$. We now obtain $e_{i+1}\cup a=(f_{i+1}\cup b)\cup a=a\cup b=a^*$ and by the modular law $e_{i+1}\cap a=(f_{i+1}\cup b)\cap a=f_{i+1}\cup(b\cap a)=f_{i+1}\cup b_*=f_{i+1}$. This verifies that e_{i+1}/f_{i+1} is a lower transpose of a^*/a as well as an upper transpose of b/b_*, which is condition (C). If e_{i+1}/f_{i+1} were rather a lower covering transpose of e_i/f_i, (C) follows by a dual argument. The basis of the induction is now complete, and e/f satisfies (C).

(D) implies (F). Let x be an arbitrary join-irreducible in A^*: $x\subseteq a^*$.

Let g be a minimal element of a^*/b_* such that $x \subseteq g$. By hypothesis $a^* \supseteq g = (g \cup a) \cap (g \cup b) \supseteq (x \cup a) \cap (x \cup b) \supseteq x \cup (a \cap b) = x \cup b_* \supseteq x,\ b_*$. By minimality $g = (x \cup a) \cap (x \cup b) = x \cup (a \cap b)$. Hence x, a, b form a distributive set and $(x \cap a) \cup (x \cap b) = x \cap (a \cup b) = x \cap a^* = x$. Join-irreducibility demands $x \cap a = x$, $x \subseteq a$, $x \in A$ or $x \cap b = x$, $x \subseteq b$, $x \in B$. Hence $x \in (A + B)$, so that $A^* \subseteq A + B \subseteq A^*$ yields equality.

(F) implies (E). Let g be arbitrary in L and let $r = (a \cup b) \cap g$, $s = a \cap g, t = b \cap g, v = (a \cap g) \cup (b \cap g) = s \cup t$. Then $R = A^* \cdot G = (A + B) \cdot G = A \cdot G + B \cdot G = S + T \subseteq V \subseteq R$, since $v \subseteq r$. Hence $R = V$ and $r = v$.

(E) implies (D). For arbitrary $g \in a^*/b_*$ $g = g \cup b_* = g \cup (a \cap b) = (g \cup a) \cap (g \cup b)$, which is the necessary and sufficient condition, cited earlier, for the desired direct product condition.

(F) implies (G). $A^* = A + B$ implies $A^* - A = B - A \cdot B = B - B_*$.

(G) implies (F). $A^* = A + (B - B_*) = A + (B - A \cdot B) = A + B$.

(G) implies (H). We already have shown the equivalence of (A)–(G) inclusive. By (C) and (F) we obtain $E = E \cdot A^* = E \cdot (A + B) = E \cdot A + E \cdot B = F + B$. Therefore $E - F = B - B \cdot F = B - B_* = A^* - A$ for all $e/f \in Q$.

(H) implies (G) trivially. This completes the proof of Theorem 4.

THEOREM 5. *A modular lattice L is distributive if and only if each $a \in L$ has exactly one upper (lower) complement.*

PROOF. First, suppose each $a \in L$ has exactly one upper complement. Assume that L is nondistributive. It will then have a nondistributive modular sublattice of order 5 with coverings: $c > e_1$, e_2, $e_3 > d$ for distinct e_1, e_2, e_3. Let $c \cup a/a$ be a maximal upper transpose of the prime quotient c/e_1 and therefore a maximal prime quotient of the complete set Q of projective prime (trivially complemented) quotients to which the prime quotients of c/d belong. By Lemma 1 $c \cup a = a^* > a$. Let $b/b \cap d$ be a minimal lower transpose of e_2/d hence a minimal lower transpose of $a \cup c/a$. Thus $b = a'_U$, the unique upper complement of a, and $b/b \cap d$ is $b/a \cap b$. Likewise a minimal lower transpose of e_3/d must be $b/a \cap b$. But then $e_2 = b \cup d = e_3$, a contradiction. Hence L is distributive.

Conversely, suppose L is distributive. Consider arbitrary $a \in L$ with $b = a'_U$, $a = b'_L$. Condition (E) of Theorem 4 holds, hence also conditions (A) and (B) by that Theorem.

DEFINITION 2. The quotient a/b of a modular lattice L is called a central sublattice of L if and only if a/b is complemented and has no upper and no lower transposes.

THEOREM 6. *In a modular lattice L a/b is a central sublattice if and only if $a=b^*$ and $b=a^*$.*

PROOF. This follows directly from Definition 2 and Lemma 1.

THEOREM 7. *In a modular lattice L if b is both an upper and a lower complement of a then $a^*=b^*$, $a_*=b_*$, and a^*/b_* is a central sublattice.*

PROOF. By Theorem 1 a is also both an upper and a lower complement of b. Hence $a^*=a\cup b=b^*$ and $b_*=a\cap b=a_*$. Both b/b_* and a/a_* are complemented. By [2, p. 105, Theorem 6, L7′] a and b are both joins of elements covering $a_*=b_*$, therefore, so is $a^*=b^*$, and by the same Theorem a^*/a_* is complemented. If a^*/a_* has an upper transpose, so would also the subquotient a^*/a by Dedekind's Transposition Principle, [2, Theorem 6, p. 73] violating the maximality asserted by Lemma 1. Thus a^*/a_* has no upper transpose, and by a dual argument has no lower transpose. Hence a^*/a_* is central.

THEOREM 8. *For an arbitrary element a of a complemented modular lattice L, b is an ordinary complement of a if and only if b is an upper (lower) complement of a.*

PROOF. All quotients of L are complemented. Hence $a^*=u=b^*$ and $a_*=z=b_*$. The equivalence then follows directly from application of the definitions of each of the types of complements.

Theorem 9 and its corollary are decidedly stronger than the converse of Theorem 8.

THEOREM 9. *If there exists one element a of a modular lattice L for which $b\in L$ is simultaneously an upper, a lower, and an ordinary complement of a, then L is complemented.*

PROOF. By Theorem 7 $a^*=a\cup b=u$, $a_*=a\cap b=z$, and $a^*/a_*=u/z=L$ is complemented.

COROLLARY. *If in a modular lattice $a_*=z$ ($a^*=u$) and b is both an ordinary and an upper (a lower) complement of a, then L is complemented.*

We note that $a_*=z$ whenever $a=z$ or $a>z$.

BIBLIOGRAPHY

1. S. P. Avann, *Dual symmetry of projective sets in a finite modular lattice*, Trans. Amer. Math. Soc. vol. 89 (1958) pp. 541–558.

2. G. Birkhoff, *Lattice theory*, rev. ed., Amer. Math. Soc. Colloquium Publications, vol. 25, 1948.

UNIVERSITY OF WASHINGTON