MODULARITY RELATIONS IN LATTICES

R. J. MIHALEK

1. Introduction. Linear independence has been formulated lattice-theoretically by G. Birkhoff [1], J. von Neumann [4] and, in particular, L. R. Wilcox [5], who studied it in connection with ordinary modularity considered as a binary relation. In this work, the concept of a modularity relation is defined abstractly from which the theory of independence is developed. These results generalize those of S. Maeda [2] whose abstraction of independence characterizes ordinary independence. Also quasi-modularity relations are considered abstractly, which relations arise in the theory of quasi-dual-ideals [7]. Relations studied earlier by the author [3] are shown to be instances of the abstract relations considered here.

Throughout this paper L is to be a lattice with order \( \leq \), join + and meet \( \cdot \). For \( b, c \in L \), \((b, c)M\) (read \((b, c)\) modular) means \((a+b)c = ac + bc\) for every \( a \leq c \) \((M\) will be referred to as ordinary modularity).

The notations \( \subset, +, \cdot, \emptyset, \times \) are respectively set-theoretic inclusion, sum, product, the empty set and cartesian product, and the set of all elements \( x \) with the property \( E(x) \) is denoted by \([x; E(x)]\).

2. Modularity relations and independence. First, the notion of a modularity relation is defined abstractly, which is then used in the definition of the independence relation and the development of the independence theory.

(2.1) Definition. Let \( R \subset T \subset L \times L \). The relation \( R \) is a modularity relation under \( T \) means

(a) \((b, c)R, b' \leq b, c' \leq c, b'c' = bc, (b', c')T \) implies \((b', c')R\);

(b) \((c, d)R, (b, c+d)R, b(c+d) = cd \) implies \((b+c, d)R, (b+c)d = cd\).

Part (a) of the definition would be too broad for the purposes considered here if the condition \((b', c')T\) were omitted from the hypotheses. The set \( T \) is introduced merely to provide a control on the pairs that are eligible to be in \( R \) and its role will become evident in the examples considered in the subsequent sections.

(2.2) Definition. For \( R \) a modularity relation under \( T \), \( R \) is said

(a) to satisfy the intersection property if \((c, d)R, (b, c+d)R, b(c+d) = cd \) implies \((b+d)(c+d) = d\);

(b) to be symmetric at \( a \), for \( a \in L \), if \((b, c)R, bc = a \) implies \((c, b)R\).

Examples exist showing that a modularity relation does not necessarily satisfy these properties.
(2.3) Definition. Let $R$ be a modularity relation under $T$. For $n \geq 2$, $a, a_1, \ldots, a_n \in L$, $(a_1, \ldots, a_n)_R a$ (read $(a_1, \ldots, a_n)$ $R$-independent over $a$) means $(\sum_U a_i, \sum_V a_i)_R R$ for every nonempty $U, V \subset [1, \ldots, n]$ such that $j < k$ for $j \in U, k \in V$.

Throughout this section it is assumed that $R$ is a modularity relation under $T$, $n \geq 2$ and $a, a_1, \ldots, a_n \in L$.

(2.4) Corollary. Let $(a_1, \ldots, a_n)_R a$.

(a) If $a_i \neq a$ for $1 \leq i \leq n$, then $a_i \neq a$ for $i \neq j$.

(b) If $1 \leq k_1 < \cdots < k_m \leq n$, $m \geq 2$, then $(a_{k_1}, \ldots, a_{k_m})_R a$.

(c) If $a \leq a'_i \leq a_i$ for $1 \leq i \leq n$, then $(a_i', \ldots, a_n')_R a$ provided $(\sum_U a'_i, \sum_V a'_i)_T$ for every nonempty $U, V \subset [1, \ldots, n]$ such that $j < k$ for $j \in U, k \in V$.

(2.5) Theorem. If $(a_1, \ldots, a_n)_R a$, then $(a_i, a_{i+1} + \cdots + a_n)_R a$ for every $i = 1, \ldots, n-1$, and conversely, provided $(\sum_V a_i)_T$ for every nonempty $V \subset [1, \ldots, n]$.

Proof. The forward implication is immediate. The reverse is obvious for $n = 2$. Suppose it holds for $q \leq n - 1$ where $n \geq 3$. Let $(a_i, a_{i+1} + \cdots + a_n)_R a$ for $i = 1, \ldots, n-1$ and let $U, V \subset [1, \ldots, n]$ such that $U, V$ are nonempty and $j < k$ for $j \in U, k \in V$. Denote $U$ by $[j_1, \ldots, j_u]$ and $V$ by $[k_1, \ldots, k_v]$, where, without loss of generality, $j_1 < \cdots < j_u < k_1 < \cdots < k_v$. Then by (2.4.c),

$$(a_{j_i}, a_{j_i+1} + \cdots + a_{k_i})_R a$$

and

$$(a_{k_i}, a_{k_i+1} + \cdots + a_{k_v})_R a$$

for $i = 1, \ldots, u$.

In case $U + V = [1, \ldots, n]$, it follows from the induction hypothesis that $(a_{j_1}, \ldots, a_{j_u}, a_{k_1}, \ldots, a_{k_v})_R a$, whence $(\sum_V a_i)_R a$. Let $U + V = [1, \ldots, n]$. From the above argument,

$$\left( a_{j_1} + \cdots + a_{j_u}, \sum_V a_i \right)_R a,$$

and by hypotheses, $(a_{j_1}, a_{j_2} + \cdots + a_{j_u} + \sum_V a_i)_R a$. Thus (2.1.b) yields $(\sum_V a_i)_R a$. Hence the reverse implication holds for $q = n$ and the result follows by induction.

(2.6) Theorem. Let $R$ satisfy the intersection property. If $(a_1, \ldots, a_n)_R a$, then $(\sum_U a_i)(\sum_V a_i) = \sum_{UV} a_i$ for every $U, V \subset [1, \ldots, n]$ such that $UV \neq \emptyset$ and $j < k < m$ for $j \in U - UV, k \in V - UV, m \in UV$.

Proof. Let $W = U - UV, X = V - UV$. Then by the hypotheses,
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whence

(\sum_w a_i + \sum_{uv} a_i)(\sum_x a_i + \sum_{uv} a_i) = \sum_{uv} a_i \text{ by virtue of the intersection property.}

(2.7) Lemma. Let \( R \) satisfy the intersection property. If \((a_1, \ldots, a_n)Ra, U + V = [1, \ldots, n], UV = \emptyset, \) then

\( (\sum_{a_1})(\sum_{v} a_i) = (a_n)(\sum_{u} a_i)(\sum_{v} a_i). \)

Proof. Let \( U, V \not= \emptyset \) and let \( 1 \in U. \) Partition the set \([1, \ldots, n]\) with sets \( W_i \) defined so that \( W_{2i-1} \subset U, W_{2i} \subset V, \) and \( j' < k' \) for \( j' \in W_j, k' \in W_k, j < k. \) (The existence of such a partition is readily proved inductively.) Then \( 1 \in W_1 \) and for some \( m, n \in W_m. \) The result is immediate for \( m = 2; \) let \( m \geq 3. \) Define \( b_j = \sum_{w_j} a_i \) for \( 1 \leq j \leq m. \) Then

\( (\sum_{v} a_i)(\sum_{v} a_i) = (\sum_{u} a_i)(b_1 + \sum_{2} a_i)(b_2 + \sum_{3} a_i)(\sum_{v} a_i) = (\sum_{u} a_i)(\sum_{v} a_i)(\sum_{v} a_i), \) the last equality holding by virtue of the intersection property. For \( m \geq 4, \) let \( 3 \leq q < m. \) Then

\[
\left( \sum_{1}^{q-1} b_i + \sum_{q+1}^{m} b_i \right) \geq \sum_{U} a_i
\]

or \( \sum_{v} a_i \) according as \( q \) is even or odd. Thus

\[
(\sum_{U} a_i)(\sum_{q} b_i)(\sum_{V} a_i)
= (\sum_{U} a_i)(\sum_{1}^{q-1} b_i + \sum_{q+1}^{m} b_i)(b_q + \sum_{q+1}^{m} b_i)(\sum_{V} a_i)
= (\sum_{U} a_i)(\sum_{q+1}^{m} b_i)(\sum_{V} a_i).
\]

Therefore

\[
(\sum_{U} a_i)(\sum_{V} a_i) = (\sum_{U} a_i)(b_m)(\sum_{V} a_i).
\]

Let \( X = W_m - [n] \) with \( X \not= \emptyset; \) otherwise, the proof is complete. Then

\[
(\sum_{1}^{m-1} b_i + a_n) \geq \sum_{U} a_i \text{ or } \sum_{V} a_i \text{ according as } m \text{ is even or odd, whence}
\]

\[
(\sum_{U} a_i)(b_m)(\sum_{V} a_i)
= (\sum_{U} a_i)(\sum_{1}^{m-1} b_i + a_n)(\sum_{X} a_i + a_n)(\sum_{V} a_i)
= (\sum_{U} a_i)(a_n)(\sum_{V} a_i).
\]
Hence
\[
\left( \sum_U a_i \right) \left( \sum_V a_i \right) = (a_n) \left( \sum_U a_i \right) \left( \sum_V a_i \right).
\]

(2.8) Theorem. Let \( R \) satisfy the intersection property. If \((a_1, \ldots, a_n) \in R_a\), then for nonempty disjoint \( U, V \subseteq [1, \ldots, n]\),
\[
(\sum_U a_i)(\sum_V a_i) = a.
\]

Proof. The result is immediate for \( n = 2 \). Suppose it holds for \( g \leq n-1 \) where \( n \geq 3 \). Then it holds for \( U + V \neq [1, \ldots, n] \), with an application of (2.4.b). Let \( U + V = [1, \ldots, n] \). From the lemma,
\[
(\sum_U a_i)(\sum_V a_i) = (a_n)(\sum_U a_i)(\sum_V a_i).
\]
Let \( n \in V \). Then for \( V = [n] \),
\[
(\sum_U a_i)(\sum_V a_i) = a \text{ by definition}, \quad \text{and for } V \neq [n], \quad (\sum_U a_i)(\sum_V a_i) = (a_n)(\sum_U a_i)(\sum_V a_i) = a \sum_V a_i = a \text{ by the induction hypothesis}.
\]
Similarly, for \( n \in U \),
\[
(\sum_U a_i)(\sum_V a_i) = a \text{ by the induction hypothesis}.
\]
Hence the result holds for \( n = q \) and the proof is complete.

(2.9) Definition. Define \((a_1, \ldots, a_n) \in R_a \) (read \((a_1, \ldots, a_n) \text{ symmetrically } R\text{-independent over } a\)) to mean \((a_n, \ldots, a_1) \in R_a \) for every permutation \((i_1, \ldots, i_n)\) of the integers \([1, \ldots, n]\).

(2.10) Corollary. (a) The relation \( R_a \) is symmetric. (b) If \((a_1, \ldots, a_n) \in R_a \), then \((a_1, \ldots, a_n) \in R_a \).

(2.11) Theorem. If \((a_1, \ldots, a_n) \in R_a \), then \((a_j, \sum_{i \neq j} a_i) \in R_a \) for \( 1 \leq j \leq n \), and conversely, provided \((a_j, \sum_V a_i) \in T \) for every nonempty \( V \subseteq [1, \ldots, n] \) such that \( j \notin V \).

Proof. This follows from (2.5) in a manner similar to the corresponding result in [5].

(2.12) Theorem. Let \( R \) satisfy the intersection property. If \((a_1, \ldots, a_n) \in R_a \), then \((\sum_U a_i)(\sum_V a_i) = \sum_{UV} a_i \) for every \( U, V \subseteq [1, \ldots, n] \) such that \( UV \neq \emptyset \).

Proof. Let \( U \subseteq V \) and \( V \subseteq U \). Then let \( U - UV = [i_1, \ldots, i_u] \), \( V - UV = [j_1, \ldots, j_v] \), \( UV = [k_1, \ldots, k_w] \), where the \( i_m, j_m \) and \( k_m \) are distinct. Define
\[
b_m = \begin{cases} 
a_{i_m} & \text{for } 1 \leq m \leq u, \\
a_{j_m-u} & \text{for } u + 1 \leq m \leq u + v, \\
a_{k_m-u-v} & \text{for } u + v + 1 \leq m \leq u + v + w. 
\end{cases}
\]

Then \((b_1, \ldots, b_{u+v+w}) \in R_a \) by (2.9) and (2.4.b). Also
\[
U' = [1, \ldots, u, u + v + 1, \ldots, u + v + w]
\]
and

\[ V' = [u + 1, \ldots, u + v + w] \]

satisfy the hypotheses of (2.6), whence

\[
\left( \sum_{U} a_i \right) \left( \sum_{V} a_i \right) = \left( \sum_{U'} b_i \right) \left( \sum_{V'} b_i \right) = \sum_{U'V'} b_i = \sum_{U'V'} a_i.
\]

In the remainder of this section, some results are stated for \( R \) symmetric at \( a \). The proofs of these results are similar to those of the corresponding results in [5] and will be omitted. In case \( R \) were a symmetric relation, it is evident that \( R \) would be symmetric at \( a \) for every \( a \in L \). If \( R \) is symmetric at \( a \), then the relation \( R_a \) is symmetric, or equivalently, \((b, c)R_a \) if and only if \((b, c)\overline{R}_a \).

(2.13) **Lemma.** Let \( R \) be symmetric at \( a \). If \((c, b, d)R_a \), then \((b, c, d)R_a \).

(2.14) **Theorem.** If \( R \) is symmetric at \( a \), then \((a_1, \ldots, a_n)\overline{R}_a \) if and only if \((a_1, \ldots, a_n)R_a \).

(2.15) **Corollary.** If \( R \) is symmetric at \( a \), then \((a_1, \ldots, a_n)R_a \) if and only if \(( \sum_{U} a_i, \sum_{V} a_i )R_a \) for every nonempty disjoint \( U, V \subset \{1, \ldots, n\} \).

(2.16) **Theorem.** Let \( R \) be symmetric at \( a \) and let \( b_1, \ldots, b_m \in L \) where \( m \geq 2 \). If \((a_1, \ldots, a_n)R_a \), \((b_1, \ldots, b_m)R_a \) and \((\sum_{U} a_i, \sum_{V} b_i)R_a \), then \((a_1, \ldots, a_n, b_1, \ldots, b_m)R_a \).

(2.17) **Corollary.** Let \( R \) be symmetric at \( a \) and for \( j = 1, \ldots, n \), let \( m_j \geq 2 \) and \( a_{ij} \in L \) for \( i = 1, \ldots, m_j \). If \((a_{ij}, \ldots, a_{m_{ij}})R_a \) for \( j = 1, \ldots, n \) and if \((\sum_{U} a_{1i}, \ldots, \sum_{U} a_{ni})R_a \), then

\[
(a_{11}, \ldots, a_{m_{11}}, \ldots, a_{1n}, \ldots, a_{mn})R_a.
\]

3. **Quasi-modularity relations.** In the study of quasi-dual-ideals, the relations of weak modularity, as denoted by Wilcox [7], and quasi-modularity, as denoted by the author [3], arise with properties similar to those of ordinary modularity. In this section the material of §2 is applied in an abstraction of these relations.

(3.1) **Definition.** A nonempty subset \( S \) of \( L \) is a quasi-dual-ideal (q.d.i.) if

(a) \( x \in S \), \( y \geq x \) implies \( y \in S \);
(b) \( x, y \in S \), \((x, y)M \) implies \( xy \in S \).

The smallest q.d.i. containing a set \( T \) (or elements \( a, b, c, \ldots \)) is denoted by \( \{T\} \) (or \( \{a, b, c, \ldots\} \)). The set of all q.d.i. is \( \mathcal{Q} \) and the set of all principal q.d.i. (of the form \( \{a\} \)) is \( \mathcal{S} \). For \( \alpha, \beta \in \mathcal{Q} \),
\( \alpha \leq \beta \) means \( \alpha \supset \beta \), \( \alpha \cup \beta = \alpha \cdot \beta \) and \( \alpha \cap \beta = \{\alpha + \beta\} \).

It is useful to note that the principal q.d.i. of \( L \) coincide with the principal dual ideals of \( L \). For the next corollary and for all statements with reference to \( L \) in the remainder of the paper, it is assumed that \( \text{l.u.b.} \ L = 1 \) exists.

(3.2) Corollary. The set \( L \) is a complete lattice with respect to \( \leq \); the lattice operations are \( \cup \), \( \cap \), and \( L \) and \( \{1\} \) are the zero and unit respectively. If \( (b, c)M, \ (b, c) = \{bc\} \). The lattice \( L \) is isomorphic to the set \( S \), a lattice subset (not necessarily a sublattice) of \( L \), under \( a \rightarrow \{a\} \).

Proof. In \( S \), \( \text{l.u.b.} \ [\{a\}, \{b\}] = \{a+b\} \) and \( \text{g.l.b.} \ [\{a\}, \{b\}] = \{ab\} \). The isomorphism now follows and the remainder is immediate.

(3.3) Definition. Let \( Q \subset L \times L \). Then \( Q \) is a quasi-modularity relation means that \( Q = [(\{b\}, \{c\}); (b, c)Q] \) is a modularity relation under \( S \times S \) in \( L \). For \( Q \) a quasi-modularity relation, \( Q \) is said to satisfy the intersection property (to be symmetric at \( \alpha \), for \( \alpha \in L \)) if \( Q \) satisfies the intersection property (if \( Q \) is symmetric at \( \alpha \)) in \( L \).

(3.4) Definition. Let \( Q \) be a quasi-modularity relation. For \( n \geq 2, a_1, \ldots, a_n \in L \) and \( \alpha \in L \), \( (a_1, \ldots, a_n)Q_{\alpha} \) (read \( (a_1, \ldots, a_n) \) \( Q \)-quasi-independent over \( \alpha \)) means \( (\{a_1\}, \ldots, \{a_n\})Q_{\alpha} \) where \( Q \) is defined as in (3.3).

(3.5) Corollary. If \( (a_1, \ldots, a_n)Q_{\alpha} \), then \( \sum_{U} a_i, \sum_{V} a_i \) \( Q \), \( \{\sum_{U} a_i, \sum_{V} a_i\} = \alpha \) for every nonempty \( U, V \subset [1, \ldots, n] \) such that \( j < k \) for \( j \in U, k \in V \), and conversely.

The corollary shows the analogy between \( Q \)-quasi-independence over a q.d.i. of \( L \) and \( R \)-independence over an element of \( L \) as defined in (2.3). The results of the independence theory of the previous section may be applied to \( Q \), yielding a corresponding theory for \( Q \). If one keeps in mind the equalities \( \{b\} \cup \{c\} = \{b+c\}, \{b\} \cap \{c\} = \{b, c\} \) and that \( \alpha \leq \{a\} \) means \( a \in \alpha \), the independence theory for \( Q \) may be stated free of the notation of the lattice \( L \).

4. Examples. An example of a modularity relation is obtained from a special case of relative modularity, the latter being a relativization of ordinary modularity.

(4.1) Definition. For \( S \subset L, b, c \in L \), \( (b, c)M_S \) (read \( (b, c) \) modular relative to \( S \)) means \( (a+b)c = a+bc \) for every \( a \in S \) such that \( a \leq c \).

Evidently, \( M = M_L \). In addition, \( M_S \) satisfies many of the properties of \( M \), some in a modified form. In particular, the next lemma is of interest.
(4.2) **Lemma.** If \((b, c)M_S, b' \leq b, c' \leq c, b'c' = bc\), then \((b', c')M_S\).

**Proof.** Let \(a \leq c'\), \(a \in S\). Then \((a+b')c' \leq (a+b)c = a+bc = a+b'c'\), whence \((b', c')M_S\) since the reverse inequality \((a+b')c' \geq a+b'c'\) holds universally for \(a \leq c'\).

(4.3) **Theorem.** If \(S\) is join-closed, then \(R = (S \times L) \cdot M_S\) is a modularity relation under \(S \times L\).

**Proof.** Part (a) of (2.1) readily follows with an application of (4.2). For Part (b), let \((c, d)R, (b, c+d)R, b(c+d) = cd\). Then \(b, c \in S\), \((b+c, d) \in S \times L\) and \(b(c+d) \leq c\). Now let \(a \leq d, a \in S\). Then \(a+c \in S\), \(a+c \leq c+d\) and

\[
(a + (b + c))d = ((a + c) + b)(c + d) = ((a + c) + b(c + d))d = (a + (c + b(c + d)))d = (a + c)d = a + cd \leq a + (b + c)d.
\]

Thus \((b+c, d)M_S\), whence \((b+c, d)R\). Also

\[
(b + c)d = (c + b)(c + d)d = (c + b(c + d))d = cd.
\]

(4.4) **Theorem.** If \(S\) is join-closed, then \(R = (S \times S) \cdot M_S\) is a modularity relation under \(S \times S\) satisfying the intersection property.

**Proof.** The proof that \(R\) is a modularity relation under \(S \times S\) is essentially the proof of (4.3). For the remainder, let \((c, d)R, (b, c+d)R, b(c+d) = cd\). Then \(d \in S\), \(b(c+d) \leq d\) and since \((b, c+d)M_S, (b+d) \cdot (c+d) = d+b(c+d) = d\).

Two examples of quasi-modularity relations are now considered.

(4.5) **Definition.** For \(b, c \in L\),

(a) \((b, c)M_0\) (read \((b, c)\) weakly modular) means \(\{a+b, c\} = \{a\}\) \(\cup \{b, c\}\) for every \(a \leq c\);

(b) \((b, c)M_q\) (read \((b, c)\) quasi-modal) means \((b, c)M_S\) where \(S = \{b, c\}\).

(4.6) **Theorem.** The relations \(M_0\) and \(M_q\) are quasi-modularity relations satisfying the intersection property.

The proof of this theorem is omitted. It is of interest to note that always \(M_0 \subset M_q\) and that examples of left-complemented \([6]\) lattices exist for which the inclusion is proper.

To show that the notion of a modularity relation is more general than ordinary modularity, one may consider the relation \(Q\) in \(L\) corresponding to \(M_0\), which is incidentally \((S \times S) \cdot M_S\). In case \(L\) is not a modular lattice, this \(Q\), although a modularity relation, is not ordinary modularity for \(L\).
REFERENCES


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