HAUSDORFF TRANSFORMS OF BOUNDED SEQUENCES

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Let \( \phi \) be a complex valued function of bounded variation on \([0, 1]\) such that \( \phi(0) = 0 \). The statement that \( H(\phi) \) is the Hausdorff transformation generated by \( \phi \) means that if \( x \) is a complex number sequence, then \( y = H(\phi)x \) where

\[
v_n = \int_0^1 \sum_{p=0}^{n} C_{n,p} t^p (1 - t)^{n-p} x_p d\phi(t) \quad (n = 0, 1, 2, \ldots).
\]

The sequence is said to be \( H(\phi) \) evaluable to \( L \) if \( y_n \to L \) as \( n \to \infty \). The transformation \( H(\phi) \) is regular (i.e., \( \lim x_n = L \) implies \( \lim y_n = L \)) if and only if \( \phi(0+) = 0 \) and \( \phi(1) = 1 \). If \( x \) is a complex number sequence, then \( C(x) \) denotes the set of limit points of \( x \).

Among the regular Hausdorff transformations are the methods \( C_r \) (\( \Re r > 0 \)) of Cesàro, \( H_r \) (\( \Re r > 0 \)) of Hölder and \( E_r \) (\( 0 < r < 1 \)) of Euler for which \( \phi(t) = 1 - (1 - t)^r \),

\[
\phi(t) = \begin{cases} 
1 & 0 \leq t < r, \\
\frac{1}{r} \int_0^t \log \left( \frac{1}{u} \right) u^{-r} du & r \leq t < 1
\end{cases}
\]

respectively. It has been shown by Barone [2] that the regular Hausdorff transformations \( C_r \) (\( \Re r > 0 \)), \( H_r \) (\( r \) a positive integer) and \( E_r \) (\( 0 < r < 1 \)) have the property that the set of limit points of the transform of each bounded complex number sequence is connected. In each case the result was obtained by use of

**Theorem A.** If \( y \) is a bounded complex number sequence such that \( |y_n - y_{n-1}| \to 0 \) as \( n \to \infty \), then \( C(y) \) is connected.

The object of this note is to prove the following

**Theorem.** If \( H(\phi) \) is a regular Hausdorff transformation, then the following two statements are equivalent:

(i) \( \phi(1-) = \phi(1) \),

(ii) \( C(H(\phi)x) \) is connected for each bounded complex number sequence \( x \).

1. **Proof that** (i) **implies** (ii). If \( x \) is a bounded complex number sequence, then \( y = H(\phi)x \) is bounded and for each positive integer \( n \)

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| y_n - y_{n-1} |
\leq M \int_0^1 \sum_{p=0}^n | C_{n,p} (1 - t)^{n-p} - C_{n-1,p} (1 - t)^{n-1-p} | d\phi(t) |

where \( M = \text{l.u.b. } |x_p| \) and \( C_{n-1,n} t^n (1 - t)^{-1} = 0 \) \((0 \leq t \leq 1)\). If \( b_n(t) \) denotes the largest integer which does not exceed \( nt \), we have \([2, p. 751]\)

\[ | y_n - y_{n-1} | \leq 2M \int_0^1 C_{n-1,b_n(t)} t^{b_n(t)+1} (1 - t)^{n-1-b_n(t)} | d\phi(t) | . \]

Let \( \epsilon > 0 \). Since \( \phi(0^+) = \phi(0) \) and \( \phi(1-) = \phi(1) \) we can choose \( \delta > 0 \) so that

\[ 2M \int_0^\delta | d\phi(t) | < \epsilon/3 \quad \text{and} \quad 2M \int_{1-\delta}^1 | d\phi(t) | < \epsilon/3. \]

An application of Stirling's formula shows that there exists a positive number \( k \) such that, if \( 1 < n\delta, \)

\[ C_{n-1,b_n(t)} t^{b_n(t)+1} (1 - t)^{n-1-b_n(t)} < k \frac{(n - 1)^{n-1/2}}{(n - 1/\delta)^n} \quad (\delta \leq t \leq 1 - \delta). \]

Consequently, if \( v \) denotes the total variation of \( \phi \) on \([0, 1]\),

\[ | y_n - y_{n-1} | \leq \epsilon/3 + 2Mvk \frac{(n - 1)^{n-1/2}}{(n - 1/\delta)^n} + \epsilon/3 < \epsilon \]

for \( n \) sufficiently large. Thus \( | y_n - y_{n-1} | \to 0 \) as \( n \to \infty \) so that, by Theorem A, \( C(y) \) is connected. Hence (i) implies (ii).

2. **Proof that (ii) implies (i).** Suppose there exists a regular Hausdorff transformation \( H(\phi) \) with \( \phi(1-) \neq \phi(1) \) such that \( C(H(\phi)x) \) is connected for each bounded complex number sequence \( x \). It is sufficient to assume \( \phi \) real valued. Let \( \phi = \phi_1 + \phi_2 \) where

\[ \phi_1(t) = \begin{cases} \phi(t), & 0 \leq t < 1, \\ \phi(1-), & t = 1, \end{cases} \]

\[ \phi_2(t) = \begin{cases} 0, & 0 \leq t < 1, \\ \phi(1) - \phi(1-), & t = 1 \end{cases} \]

and for each pair \( n, p \) of nonnegative integers, \( p \leq n \), let

\[ A_{np} = \int_0^1 C_{n,p} (1 - t)^{n-p} d\phi_1(t). \]

It is not difficult to show \([3, pp. 308-309]\) that (1) \( A_{np} \to 0 \) as \( n \to \infty \) \((p = 0, 1, 2, \cdots)\). If \( n \) is a nonnegative integer and \( \epsilon > 0 \), then
\[ |A_{n+p,p}| \leq (n + p)^n \int_0^\delta t^p |d\phi_1(t)| + \int_\delta^1 |d\phi_1(t)| \leq (n + p)^n \delta^p \int_0^\delta |d\phi_1(t)| + \int_\delta^1 |d\phi_1(t)| < \varepsilon \]

for \( p \) sufficiently large, provided \( \delta \) is a number in \((0, 1)\) so chosen that

\[ \int_\delta^1 |d\phi_1(t)| < \varepsilon/2. \]

Hence (2) \( A_{n+p,p} \to 0 \) as \( p \to \infty \) \((n = 0, 1, 2, \cdots )\). Suppose each of \( n \) and \( p \) is a positive integer. Since the function \( C_{n+p,p} t^p (1 - t)^n \) is monotone on each of the intervals \([0, p/(n+p)]\) and \([p/(n+p), 1]\) we have, after an integration by parts, the inequality

\[ |A_{n+p,p}| \leq \int_0^{p/(n+p)} |\phi_1(t)| \ d[C_{n+p,p} t^p (1 - t)^n] \]

\[ + \int_{p/(n+p)}^1 |\phi_1(t)| \ d[-C_{n+p,p} t^p (1 - t)^n] \]

\[ \leq 2MC_{n+p,p} p/(n+p)^p [n/(n+p)]^n < k(1/n + 1/p)^{1/2} \]

where \( M = \text{l.u.b.}_{0 \leq t \leq 1} |\phi_1(t)| \) and \( k \) is a constant arising from an application of Stirling’s formula.

From conditions (1), (2) and (3) it follows that \( A_{n+p} \to 0 \) as \( n \to \infty \) uniformly with respect to \( p \). According to a result of Agnew [1] this property of the numbers \( A_{n+p} \) is sufficient to ensure the existence of a divergent sequence \( x \) of zeros and ones such that \( H(\phi_1)x \) has limit zero. Therefore \( y = H(\phi)x = H(\phi_1)x + H(\phi_2)x \) is the sum of a sequence with limit zero and a divergent sequence each element of which is zero or \( \phi_2(1) \). Consequently, \( C(y) \) is not connected. This is a contradiction. Hence (ii) implies (i).

References


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