HAUSDORFF TRANSFORMS OF BOUNDED SEQUENCES

J. H. WELLS

Let $\phi$ be a complex valued function of bounded variation on $[0, 1]$ such that $\phi(0) = 0$. The statement that $H(\phi)$ is the Hausdorff transformation generated by $\phi$ means that if $x$ is a complex number sequence, then $y = H(\phi)x$ where

$$v_n = \int_0^1 \sum_{p=0}^n C_{n,p} t^p (1 - t) \ n^{-p} x_p d\phi(t) \quad (n = 0, 1, 2, \cdots).$$

The sequence is said to be $H(\phi)$ evaluable to $L$ if $y_n \rightarrow L$ as $n \rightarrow \infty$. The transformation $H(\phi)$ is regular (i.e., $\lim x_n = L$ implies $\lim y_n = L$) if and only if $\phi(0+) = 0$ and $\phi(1) = 1$. If $x$ is a complex number sequence, then $C(x)$ denotes the set of limit points of $x$.

Among the regular Hausdorff transformations are the methods $C_r$ (Re $r > 0$) of Cesàro, $H_r$ (Re $r > 0$) of Hölder and $E_r$ ($0 < r < 1$) of Euler for which $\phi(t) = 1 - (1-t)^r$,

$$\phi(t) = \frac{1}{\Gamma(r)} \int_0^t (\log 1/u)^{r-1} du,$$  

respectively. It has been shown by Barone [2] that the regular Hausdorff transformations $C_r$ (Re $r > 0$), $H_r$ ($r$ a positive integer) and $E_r$ ($0 < r < 1$) have the property that the set of limit points of the transform of each bounded complex number sequence is connected. In each case the result was obtained by use of

**Theorem A.** If $y$ is a bounded complex number sequence such that $|y_n - y_{n-1}| \rightarrow 0$ as $n \rightarrow \infty$, then $C(y)$ is connected.

The object of this note is to prove the following

**Theorem.** If $H(\phi)$ is a regular Hausdorff transformation, then the following two statements are equivalent:

(i) $\phi(1-) = \phi(1)$,

(ii) $C(H(\phi)x)$ is connected for each bounded complex number sequence $x$.

1. **Proof that** (i) **implies** (ii). If $x$ is a bounded complex number sequence, then $y = H(\phi)x$ is bounded and for each positive integer $n$
\[ |y_n - y_{n-1}| \leq M \int_0^1 \sum_{p=0}^n |C_n,p\rho_p(1 - t)^{n-p} - C_{n-1,p}\rho_p(1 - t)^{n-1-p}| \, |d\phi(t)| \]

where \(M = \text{l.u.b.} |x_p|\) and \(C_{n-1,n}\rho^n(1 - t)^{-1} = 0\) \((0 \leq t \leq 1)\). If \(b_n(t)\) denotes the largest integer which does not exceed \(nt\), we have [2, p. 751]

\[ |y_n - y_{n-1}| \leq 2M \int_0^1 C_{n-1,b_n(t)}\rho^{b_n(t)+1}(1 - t)^{n-1-b_n(t)} \, |d\phi(t)| . \]

Let \(\epsilon > 0\). Since \(\phi(0^+) = \phi(0)\) and \(\phi(1^-) = \phi(1)\) we can choose \(\delta > 0\) so that

\[ 2M \int_0^\delta |d\phi(t)| < \epsilon/3 \quad \text{and} \quad 2M \int_{1-\delta}^1 |d\phi(t)| < \epsilon/3. \]

An application of Stirling's formula shows that there exists a positive number \(k\) such that, if \(1 < n\delta,\)

\[ C_{n-1,b_n(t)}\rho^{b_n(t)+1}(1 - t)^{n-1-b_n(t)} < k \frac{(n - 1)n^{-1/2}}{(n - 1/\delta)^n} (\delta \leq t \leq 1 - \delta). \]

Consequently, if \(v\) denotes the total variation of \(\phi\) on \([0, 1] , \)

\[ |y_n - y_{n-1}| \leq \epsilon/3 + 2Mvk \frac{(n - 1)n^{-1/2}}{(n - 1/\delta)^n} + \epsilon/3 < \epsilon \]

for \(n\) sufficiently large. Thus \(|y_n - y_{n-1}| \to 0\) as \(n \to \infty\) so that, by Theorem A, \(C(y)\) is connected. Hence (i) implies (ii).

2. Proof that (ii) implies (i). Suppose there exists a regular Hausdorff transformation \(H(\phi)\) with \(\phi(1^-) \neq \phi(1)\) such that \(C(H(\phi)x)\) is connected for each bounded complex number sequence \(x\). It is sufficient to assume \(\phi\) real valued. Let \(\phi = \phi_1 + \phi_2\) where

\[ \phi_1(t) = \begin{cases} \phi(t), & 0 \leq t < 1, \\ \phi(1^-) & t = 1, \end{cases} \quad \phi_2(t) = \begin{cases} 0, & 0 \leq t < 1, \\ \phi(1) - \phi(1^-), & t = 1 \end{cases} \]

and for each pair \(n, p\) of nonnegative integers, \(p \leq n\), let

\[ A_{np} = \int_0^1 C_n,p\rho_p(1 - t)^{n-p} \, d\phi_1(t). \]

It is not difficult to show [3, pp. 308–309] that (1) \(A_{np} \to 0\) as \(n \to \infty\) \((p = 0, 1, 2, \cdots)\). If \(n\) is a nonnegative integer and \(\epsilon > 0\), then
\[ |A_{n+p,p}| \leq (n + p)^n \int_0^\delta t^p |d\phi_1(t)| + \int_\delta^1 |d\phi_1(t)| \]

\[ \leq (n + p)^n \delta^p \int_0^\delta |d\phi_1(t)| + \int_\delta^1 |d\phi_1(t)| < \epsilon \]

for \( p \) sufficiently large, provided \( \delta \) is a number in \((0, 1)\) so chosen that

\[ \int_\delta^1 |d\phi_1(t)| < \epsilon/2. \]

Hence (2) \( A_{n+p,p} \to 0 \) as \( p \to \infty \) \((n = 0, 1, 2, \cdots)\). Suppose each of \( n \) and \( p \) is a positive integer. Since the function \( C_{n+p,p}(1-t)^n \) is monotone on each of the intervals \([0, p/(n+p)]\) and \([p/(n+p), 1]\) we have, after an integration by parts, the inequality

\[ \int_0^{p/(n+p)} |\phi_1(t)| \ d[C_{n+p,p}(1-t)^n] \]

\[ + \int_{p/(n+p)}^1 |\phi_1(t)| \ d[-C_{n+p,p}(1-t)^n] \]

\[ \leq 2MC_{n+p,p}[p/(n+p)]^p[n/(n + p)]^n < k(1/n + 1/p)^{1/2} \]

where \( M = \text{l.u.b.}_{0 \leq t \leq 1} |\phi_1(t)| \) and \( k \) is a constant arising from an application of Stirling's formula.

From conditions (1), (2) and (3) it follows that \( A_{n,p} \to 0 \) as \( n \to \infty \) uniformly with respect to \( p \). According to a result of Agnew [1] this property of the numbers \( A_{n,p} \) is sufficient to ensure the existence of a divergent sequence \( x \) of zeros and ones such that \( H(\phi_1)x \) has limit zero. Therefore \( y = H(\phi)x = H(\phi_1)x + H(\phi_2)x \) is the sum of a sequence with limit zero and a divergent sequence each element of which is zero or \( \phi_2(1) \). Consequently, \( C(y) \) is not connected. This is a contradiction. Hence (ii) implies (i).

References


The University of North Carolina