SUMS OF STATIONARY RANDOM VARIABLES

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A sequence \( x(t) (\infty < t < \infty, t \text{ an integer}) \) of elements in Hilbert space is called stationary if the inner product \( (x(t+s), x(t)) \) does not depend upon \( t \). If the Hilbert space is \( L^2 \) space with probability measure, then \( x(t) \) is a random variable and the sequence \( x(t) (\infty < t < \infty) \) is called a second-order stationary random process. Let \( X \) be the closed linear manifold spanned by all the elements of the stationary process. Then Kolmogorov [1] has shown that the equation \( x(t) U = x(t+1), \infty < t < \infty \), uniquely determines the unitary operator \( U \) with domain and range \( X \). Using the von Neumann [2] spectral representation of \( U \), we obtain the spectral representation of the random process

\[
x(t) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2\pi i u t} dx(0)E(u), \quad -\infty < t < \infty.
\]

The von Neumann [3] ergodic theorem, in the framework of Khintchine [4], is applicable, and shows that the average \( \sum_{t} x(t)/n \) converges in the mean to the random variable \( x(0)[E(0+) - E(0-) ] \) as \( n \to \infty \). In this paper we consider sums instead of averages; that is, we consider \( \sum_{t} x(t) \), and establish the following theorem.

THEOREM. Let the random variables \( x(t) (\infty < t < \infty, t \text{ an integer}) \) be a second-order stationary random process with spectral distribution function \( F(u) \). For variance \( \{ \sum_{t} x(t) \} \) to be bounded for all positive integers \( n \), each of the following two conditions is necessary and sufficient:

1. \( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{-2} u dF(u) < \infty \).

2. There is a second-order stationary random process \( y(t) (\infty < t < \infty) \) satisfying \( y(t) - y(t+1) = x(t) \).

PROOF. (NECESSARY CONDITIONS). We are given that variance \( \{ \sum_{t} x(t) \} < B \) for all positive integers \( n \). Without loss of generality we assume that the \( x(t) \) are centered so that their mean values are zero. Then

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variance \left\{ \sum_{1}^{n} x(t) \right\} = \left( \sum_{1}^{n} x(t), \sum_{1}^{n} x(t) \right).

From the spectral representation we have

\[ \sum_{1}^{n} x(t) = \int_{-.5}^{.5} e^{2\pi i u} \frac{1 - e^{2\pi i u}}{1 - e^{2\pi i u}} \, dx(0)E(u), \]

so

\[ \text{variance} \left\{ \sum_{1}^{n} x(t) \right\} = \int_{-.5}^{.5} \frac{1 - e^{2\pi i u}}{1 - e^{2\pi i u}} \, dF(u), \]

where \( F(u) = \|x(0)E(u)\|^2, \, -.5 \leq u \leq .5, \) is the spectral distribution function. Hence we have

\[ \mathcal{B} > \text{variance} \left\{ \sum_{1}^{n} x(t) \right\} = \left\{ \int_{-.5}^{0^-} + \int_{0^+}^{.5} \right\} \sin^2 \pi u \, dF(u) \]

\[ + n^2[F(0^+) - F(0^-)] \]

which shows that \( F(0^+) - F(0^-) \) must vanish. Moreover, we have

\[ \mathcal{B} > \frac{1}{\mathcal{N}} \sum_{n=1}^{\mathcal{N}} \left\{ \int_{-.5}^{0^-} + \int_{0^+}^{.5} \right\} \frac{\sin^2 \pi u}{\sin^2 \pi u} \, dF(u) \]

\[ = \left\{ \int_{-.5}^{0^-} + \int_{0^+}^{.5} \right\} \left[ \frac{1}{\mathcal{N}} \sum_{n=1}^{\mathcal{N}} \left( \frac{1}{2} - \frac{1}{2} \cos 2\pi u \right) \right] \sin^{-2} \pi u dF(u). \]

Clearly the limit of the expression in brackets, as \( \mathcal{N} \to \infty, \) is 1/2, so \( \int_{-.5}^{.5} \sin^{-2} \pi u dF(u) \) is finite. Q.E.D. (1).

The distribution function \( F(u) \) defines a Lebesgue-Stieltjes measure on the real line segment \(-.5 \leq u \leq .5.\) Let \( W \) denote the \( L^2 \) space of complex-valued measurable functions \( \Phi(u) \) defined on \(-.5 \leq u \leq .5 \) for this measure. Define a correspondence between an element \( x \) of \( X \) and an element \( \Phi(u) \) of \( W \) by

\[ x = \int_{-.5}^{.5} \Phi(u) \, dx(0)E(u) \leftrightarrow \Phi(u). \]

Then Stone [5] and Kolmogorov [1] have shown that this correspondence establishes an isomorphism between \( X \) and \( W \) that preserves inner products. The function \( e^{2\pi i u}/(1 - e^{2\pi i u}) \) belongs to \( W \) since

\[ \int_{-.5}^{.5} \left| \frac{e^{2\pi i u}}{1 - e^{2\pi i u}} \right|^2 \, dF(u) = \frac{1}{4} \int_{-.5}^{.5} \sin^{-2} \pi u d\mu < \infty. \]
If we define the element $y(t)$ of $X$ by the correspondence $y(t) \leftrightarrow e^{2\pi i ut}/(1-e^{2\pi i u})$ we see that

$$y(t) - y(t+1) \leftrightarrow \frac{e^{2\pi i ut} - e^{2\pi i u(t+1)}}{1 - e^{2\pi i u}} = e^{2\pi i ut}.$$ 

But by the spectral representation, we know that $x(t) \leftrightarrow e^{2\pi i ut}$, and hence we have $y(t) - y(t+1) = x(t)$ for all integers $t$. Since

$$(y(t + s), y(t)) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{2\pi i u(t+s)}e^{-2\pi i ut}}{1 - e^{2\pi i u}} dF(u) = \frac{1}{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i us} \sin^{-2} \pi u dF(u)$$

depends only on $s$, we see that $y(t)$ is a stationary random process. Q.E.D. (2).

**Proof.** (Sufficient conditions). Let condition (1) of the theorem be given. Since

$$\text{variance } \left\{ \sum_{1}^{n} x(t) \right\} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sin^{2} \pi u}{\sin^{2} \pi u} dF(u) \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin^{-2} \pi u dF(u)$$

we see that the variance is bounded. Q.E.D. (1).

Let condition (2) of the theorem be given. Then $\|y(t)\|$ is a finite constant. Because $\sum_{1}^{n} x(t) = y(1) - y(n+1)$ we have $\|\sum_{1}^{n} x(t)\| \leq \|y(1)\| + \|y(n+1)\|$, and so variance $\left\{ \sum_{1}^{n} x(t) \right\} = \|\sum_{1}^{n} x(t)\|^{2}$ is bounded. Q.E.D. (2).

**References**


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