COMPACT LINEAR TRANSFORMATIONS
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1. Consider a compact linear transformation $T$ (also called completely continuous transformation) from a Banach space $A$ to a Banach space $B$. Can $T$ be approximated arbitrarily close in norm by bounded linear transformations whose ranges are finite dimensional (see [1, p. 49])? The answer is affirmative for the following types of domain and range spaces: (i) both $A$ and $B$ are Hilbert spaces (see [2, p. 204]), (ii) both $A$ and $B$ are $C[0, 1]$ (see [2, p. 222] or [3]), (iii) there is no other restriction on $A$, but $B$ is of “type $A$” [4], (iv) $A$ is either $L^p$ or $C$ and there is no other restriction on $B$ (see [5, p. 536]). In this paper we shall show that the answer is also affirmative when $A$ is any Banach space and $B$ is $C(E)$, $E$ being a compact Hausdorff space.

Let $S^*$ be the strongly closed unit sphere in the conjugate space $B^*$, namely the set of all linear functionals of unit norm or less. $S^*$ is a compact Hausdorff space in the relative topology introduced in $S^*$ by the weak * topology of $B^*$ (see [1, p. 37]). For convenience, we continue to call this relative topology in $S^*$ the weak * topology. Denote by $C(S^*)$ the Banach algebra of all the complex-valued weak *-continuous functions in $S^*$. For each $x$ in $B$, the mapping $x \rightarrow x^{**}$ is an isometric isomorphism embedding $B$ as a subspace of $B^{**}$, and $x^{**} \rightarrow x^{**}$ (restricted to $S^*$) is also an isomorphism satisfying

$$||x^{**}|| = \sup_{F \in S^*} |x^{**}(F)| = ||x^{**}||_\infty;$$

where $||x^{**}||_\infty$ is the uniform norm of $x^{**}$ restricted to $S^*$. Hence we can embed $B$ as a subspace of $C(S^*)$ under the isometric isomorphism $x \rightarrow x^{**}$ (restricted to $S^*$). Consequently, by embedding $B$ in $C(S^*)$, a compact linear transformation $T$ from a Banach space $A$ to a Banach space $B$ can be approximated arbitrarily close in norm by bounded linear transformations of finite dimensional range from $A$ to $C(S^*)$. (See §3.)

The ideas in §§2 and 3 are suggested by those of Radon in [3]; (see also [2, p. 222]). Throughout this note, $C(E)$ denotes the Banach algebra of all complex-valued continuous functions defined on a compact Hausdorff space $E$, and $(T^*F)(x)$ means $(Tx)(F)$, for $x$ in $A$, $Tx$ in $C(E)$ and $F$ in $E$.

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2. In this section we shall establish the following result:
Let $T$ be a compact linear transformation from a Banach space $A$ to the space $C(E)$. Then $T^*$ is a continuous mapping on $E$ to $A^*$. (Here $A^*$ is given the usual norm topology.)

**Proof.** Take a fixed $F_0$ in $E$. To each $\epsilon > 0$, we have to show that there is an open set $O$ in $E$ containing $F_0$ such that

$$
\|T^* F - T^* F_0\| < \epsilon \quad \text{for all } F \in O.
$$

Suppose that this is not true. Then there is an $\epsilon = 2\epsilon_0 > 0$ such that (1) is satisfied by no open set $O$ in $E$ containing $F_0$. We show that it leads to contradiction.

By virtue of the compactness of $T$, the image $T(S)$ of the closed unit sphere $S$ in $A$ is separable and hence contains a sequence $\{z_n\}$ dense in the closure of $T(S)$.

The sets $U_{m,n} = \{ F \mid \|z_m(F) - z_m(F_0)\| < 1/n, F \in E \}$ are open sets in $E$ and form a sequence $\{V_k\}$. Let $O_n$ be the intersection of $V_1$, $V_2$, $\ldots$ and $V_n$. Clearly $F_0$ lies in each $O_n$ and $\{O_n\}$ is monotonic decreasing. By the supposition, for each $n$, there exists a $F_n$ in $O_n$ such that $\|T^* F_n - T^* F_0\| \geq 2\epsilon_0$. But then there is a $x_n$ in $S$ such that

$$
|\langle T x_n \rangle(F_n) - \langle T x_n \rangle(F_0)| = |\langle T^* F_n - T^* F_0 \rangle(x_n)| > \epsilon_0.
$$

Since $T$ is compact, we may suppose, by passing to a subsequence if necessary, that

$$
T x_n = y_n \text{ converges in norm to some } y \text{ in } C(E).
$$

As $y$ can be approximated arbitrarily close in norm by $\{z_n\}$, to an integer $p$ satisfying $3 < \epsilon_0 p$, there is an integer $q$ such that

$$
\|y - z_q\|_\infty < \frac{1}{p} < \frac{\epsilon_0}{3}.
$$

Let $N$ be so large that $O_N$ is contained in $U_{q,p}$. Then by (4),

$$
|y(F_n) - z_q(F_n)| < 1/p, \quad |y(F_0) - z_q(F_0)| < 1/p \quad \text{and, for } n \geq N,
$$

$$
|z_q(F_n) - z_q(F_0)| < 1/p. \quad \text{Hence}
$$

$$
|y(F_n) - y(F_0)| < 3/p < \epsilon_0 \quad \text{for } n \geq N.
$$

Now that, by virtue of (3), $|y_n(F_n) - y(F_n)|$ and $|y_n(F_0) - y(F_0)|$ both tend to zero as $n \to \infty$ and (5) together show that (2) cannot hold for all $n$. The contradiction proves that $T^*$ is continuous on $E$.

3. Let $T$ be the transformations of §2. For each $F_k$ in $E$ and each $\epsilon > 0$, the set $O_k = \{ F \mid \|T^* F - T^* F_k\| < \epsilon, F \in E \}$ is open. Let $g_k$ be a real-valued continuous function in $E$ such that $g_k = 2$ at $F_k$, $g_k = 0$
outside \( O_k, 0 \leq g_k \leq 2 \). The existence of such functions is assured by the Urysohn’s lemma. Set \( U_k = \{ F | g_k(F) > 1, F \in E \} \), then \( U_k \) is an open set containing \( F_k \), and \( U_k \subset O_k \). Since \( E \) is compact, it can be covered by some finite family of sets \( U_1, U_2, \ldots, U_n \). Setting \( h_i(F) = \inf(g_i(F), 1) \), we define inductively

\[
h_m = \inf \left( \sum_{i=1}^{m-1} h_i + g_m, 1 \right) - \sum_{i=1}^{m-1} h_i, \quad m = 2, 3, \ldots, n.
\]

The functions \( h_m \) are continuous and belong to \( C(E) \). They satisfy

\[
o \leq h_i(F) \leq 1, \quad \sum_{i=1}^{n} h_i(F) = 1 \text{ in } E,
\]

\( h_i(F) \neq 0 \) implies \( F \in O_i \). For \( x \) in \( A \), define

\[
T_n x = \sum_{i=1}^{n} (Tx)(F_i)h_i.
\]

Clearly \( T_n x \) is in \( C(E) \) and the range of \( T_n \) is finite dimensional. Using the properties of the functions \( h_i \) and the definition of \( O_i \), we can see that in \( E \)

\[
\left| (Tx)(F) - (T_n x)(F) \right| = \left| (T^*F)(x) - \sum_{i=1}^{n} h_i(F)(T^*F_i)(x) \right|
\]

\[
\leq \| T^*F - \sum_{i=1}^{n} h_i(F)T^*F_i \| \| x \|
\]

\[
\leq \sum_{i=1}^{n} h_i(F) \| T^*F - T^*F_i \| \| x \|
\]

\[
< \epsilon \| x \|.
\]

Hence

\[
\| Tx - T_n x \|_\infty < \epsilon \| x \|, \quad \| T - T_n \| \leq \epsilon.
\]

We have thus proved the following result:

A compact linear transformation \( T \) from a Banach space \( A \) to the space \( C(E) \) can be approximated arbitrarily close in norm by bounded linear transformations of finite-dimensional range.

In view of the discussion in §1, it follows that

A compact linear transformation \( T \) from a Banach space \( A \) to a Banach space \( B \), embedded in \( C(S^*) \), can be approximated arbitrarily close in norm by bounded linear transformations of finite-dimensional range from \( A \) to \( C(S^*) \).
4. If a sequence of compact linear transformations converges to a limit in norm it is known that the limit is compact. (See [1, p. 49]). In view of this property, the first result in §3 can be stated as follows:

$T$ is a compact linear transformation from a Banach space $A$ to a Banach algebra $C(E)$ if and only if $T$ can be approximated arbitrarily close in norm by bounded linear transformations of finite-dimensional range from $A$ to $C(E)$.

The method in §3 uses essentially the continuity of $T^*$. Hence from §§2 and 3 we see that

A bounded linear transformation $T$ from a Banach space $A$ to a Banach algebra $C(E)$ is compact if and only if $T^*$ is continuous from $E$ to $A^*$.

As consequences of these remarks we also see that

A linear transformation from a Banach space $A$ to a Banach space $B$ is compact if and only if when $B$ is embedded in $C(S^*)$ it can be approximated arbitrarily close in norm by bounded linear transformations of finite-dimensional range from $A$ to $C(S^*)$; and

A bounded linear transformation $T$ from a Banach space $A$ to a Banach space $B$ is compact if and only if $T^*$ is weak *-continuous on $S^*$ to $A^*$.

Let $\beta(A, B)(\beta(A, C(S^*)))$ be the Banach space of all compact linear transformations from the Banach space $A$ to the Banach space $B(C(S^*))$. We can also express the above results in the following form:

$\beta(A, B)$ can be embedded in $\beta(A, C(S^*))$. The subspace of all the transformations of finite-dimensional range in $\beta(A, C(S^*))$ is dense in $\beta(A, C(S^*))$.

When $A = B$, $\beta(A, A)$ is an algebra. To apply the results above, we can embed both the domain $A$ and range $A$ in $C(S^*)$.

We observe that the completeness of $A$ has not been used in this note.

**References**