A SUFFICIENT CONDITION FOR A MATRIC FUNCTION TO BE A PRIMARY MATRIC FUNCTION

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1. Introduction. A primary matric function is defined to be a matric function (that is, a mapping whose range and domain are sets of $n \times n$ matrices) arising from a scalar function of a complex variable. It has been shown [1] that primary matric functions are $H$-analytic. In this paper other necessary conditions for a primary matric function will be exhibited and it will then be shown that these conditions are also sufficient for a matric function to be a primary function.

We will first use a form of the definition of a primary function proposed by Frobenius and later use an equivalent form proposed by Giorgi [4]. Frobenius proposed that if the scalar function $f(z)$ is analytic at the eigenvalues of $Z$ in $\mathfrak{M}$ (the algebra of square matrices of order $n$ over the complex field) then $f(Z)$ shall be defined by

$$f(Z) = \frac{1}{2\pi i} \int_{C} \frac{f(\lambda)}{\lambda I - Z} \, d\lambda,$$

where $C$ is a set of admissible closed paths enclosing each of the distinct eigenvalues of $Z$. That is, the components of $f(Z)$ are the integrals over $C$ of the corresponding components of the matrix $f(\lambda)(\lambda I - Z)^{-1}/2\pi i$.

We wish to exhibit sufficient conditions on a matric function $F(Z)$ such that there will exist a scalar function $g(z)$ for which $F(Z) = g(Z)$ where $g(Z)$ may be computed as in (1.1).

2. Necessary conditions. It has previously been shown in [1] that primary matric functions are $H$-analytic in $\mathfrak{M}$, that is, the component functions of a primary function $g(Z)$ are analytic functions of the components $z_{ij}$ of $Z$, for $Z$ in an $\mathfrak{M}$-neighborhood of a matrix at which $g(Z)$ is defined.

If $g(z)$ is a scalar function defined at a matrix $X$, that is, $g(z)$ is analytic at the eigenvalues of $X$, and if $Y$ is such that for some non-singular matrix $P$, $Y = P^{-1}XP$, then $g$ is defined at $Y$ and $g(Y) = P^{-1}g(X)P$, as can be seen from (1.1).

If $Z$ is a matrix whose eigenvalues lie in the domain of analyticity of $g(z)$, then the $r, s$ component of $g(Z)$ is given by

1 Received by the editors January 3, 1959 and, in revised form, April 13, 1959.
2 This paper was prepared under the facilities granted by the Case Research Fund.
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\[ g(Z)_{rs} = \frac{1}{2\pi i} \int_{c} g(\lambda) (\lambda I - Z)^{-1} d\lambda, \]

where \((\lambda I - Z)^{-1}\) is the \(r, s\) component of \((\lambda I - Z)^{-1}\). For an upper triangular matrix \(Z = (z_{ij})\), \(z_{ij} = 0\) for \(i > j\), a simple computation shows that \((\lambda I - Z)^{-1}\) and thus \(g(Z)_{rs}\) depend only on the \(z_{ij}\) for which \(r \leq i \leq j \leq s\) and is zero for \(r > s\). In particular, \(g(Z)_{rr} = g(r_{rr})\) for \(Z\) a diagonal (or upper triangular) matrix.

3. Sufficient conditions. We shall now show that these necessary conditions are also sufficient. For convenience the norm of a matrix \(Z = (z_{ij})\) shall be defined by \(\|Z\| = \max_{i,j} |z_{ij}|\).

Theorem 3.1. Let \(D\) be an open domain of \(H\)-analyticity of a matric function \(F\) on \(\mathbb{M}\).

(i) Let \(F\) be such that \(X\) in \(D\) and \(Y = P^{-1}XP\) implies that \(Y\) is in \(D\) and \(F(Y) = P^{-1}F(X)P\).

(ii) Let \(F\) also be such that if \(T = (t_{ij})\), in \(D\), is a diagonal matrix, then \(F(T)_{rr}\) is a function of only \(t_{rr}\), where \(F(T)_{rr}\) is the \(r, r\) component of \(F(T)\), that is

\[ F(T)_{rr} = g_{rr}(t_{rr}). \]

Then there exists a scalar function \(g(z)\) such that for all \(Z\) in \(D\), \(g(Z) = F(Z)\).

Proof. Let \(C\) be a Jordan form for a matrix \(Z\) at which \(F\) is \(H\)-analytic, then \(C\) is a direct sum \(C_{pi} + \cdots + C_{pk}\) of canonical blocks of the form

\[
C_{pi} = \begin{bmatrix}
\lambda_i & 1 & 0 & \cdots & 0 \\
\lambda_i & 1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\lambda_i & \ddots & \ddots & 1 & \ddots \\
0 & \cdots & \cdots & \cdots & \lambda_i
\end{bmatrix}
\]

with \(p_i\) rows and columns. (The \(\lambda_i\) occurring in different \(C_{pi}\) need not be distinct.)

From (i) and Lemma 4.1 of [2] it follows that \(F(C)\) commutes with all matrices that commute with the canonical matrix \(C\). It is known that a matrix \(F(C)\) satisfying this condition must be a direct sum \(P_1(C_{pi}) + \cdots + P_k(C_{pk})\), where
and $\alpha_{im} = \alpha_{jm}$ for $\lambda_i = \lambda_j$ (see Turnbull and Aitken [7]).

Now, using a definition proposed by G. Giorgi which is equivalent to (1.1) [4] for $g(Z)$ where $g(z)$ is a scalar function, it is seen that the theorem will be proven if there exists a scalar function $g(z)$ such that, for $C = P^{-1}ZP$, where $Z$ is any matrix at which $F$ is $H$-analytic,

\[(3.2)\]

\[
\alpha_{jm} = g^{(m-1)}(\lambda_j)/(m - 1)!
\]

or,

\[(3.3)\]

\[
F(C)_{r_jr_{j+i}} = g^{(i)}(\lambda_i)/i!, \quad j = 1, \ldots, k, \quad i = 0, \ldots, p_j - 1,
\]

where $F(C)_{r_jr_{j+i}}$ is the $r_j, r_{j+i}$ component of $F(C)$ (that is, $F_{r_jr_{j+i}}$ evaluated at the components of $C$) and $r_j = 1 + \sum_{i=1}^{j-1} p_i$ for $1 < j \leq k$, $r_j = 1$ for $j = 1$. (This choice of $r_j$ proves (3.2) for components in the first row of each triangular block $P_j(C_{p_j})$ of $F(C)$ associated with $C_{p_j}$ of $C$, which is all that is necessary, since in any such block, the values on any super diagonal are all equal.)

We shall first exhibit a scalar function $g(z)$ which is determined by $F$ and then show that this function has the required property (3.3).

Let $Z$ be an arbitrary but fixed matrix such that $F$ is $H$-analytic in a neighborhood of $Z$; then $Z$ is similar to an upper triangular matrix $X = (x_{ij})$ whose eigenvalues are the $x_{ii}$. By (i) $F$ is $H$-analytic in a neighborhood of $X$. Choose any matrix $Y = (y_{ij})$ such that $y_{ij} = x_{ij}$ and $y_{ii} \neq y_{jj}$ for $i \neq j$, and $|y_{ii} - x_{ii}| < \epsilon$, where $\epsilon$ is sufficiently small such that $F$ is $H$-analytic at $Y$ (such an $\epsilon$ exists since $F$ is $H$-analytic in a neighborhood of $X$). $Y$ is similar to a diagonal matrix $A = \text{diag}(y_{kk})$ with distinct eigenvalues $y_{ii}$ and by (i) $F$ is $H$-analytic at $A$. Now, $A$ is similar to a diagonal matrix $B$ obtained from $A$ by permuting, say $y_{ii}$ and $y_{jj}$, and by (i), this same permutation is performed on $F(A)$ in order to obtain $F(B)$. Thus by (ii), $g_{ii}(y_{jj}) = F(B)_{ii} = F(A)_{jj} = g_{jj}(y_{jj})$. Hence for any $j$, $g_{jj}(z) = g_{ii}(z)$ for $i = 1, \ldots, n$ and $|z - x_{jj}| < \epsilon$ and therefore, since the $F_i$, are analytic, there exists a function $g(z) = g_{ii}(z)$, $i = 2, \ldots, n$, analytic in the open circular domains $|z - x_{jj}| < \epsilon$, $j = 1, \ldots, n$, where the $x_{jj}$ are the eigenvalues of $Z$. Thus, since $Z$ is an arbitrary matrix in $D$, there exists a function $g(z)$ which is analytic at the eigenvalues of all matrices in $D$.  

In order to show that \( g(z) \) satisfies (3.3) we first note, from (3.1), that if \( F \) is \( H \)-analytic in a neighborhood of a canonical matrix \( C \), then \( F(C) \) may be written

(3.4) \[ F(C) = \sum_{i=1}^{k} \sum_{p=0}^{p_i-1} \sum_{t=0}^{p_i-s-1} F(C)_{r_i+r_s+t} E_{p_i+s} r_i+s+t \]

where \( r_i = 1 + \sum_{j=1}^{i-1} p_j \) and \( E_{pq} \) is the matrix with a 1 in the \( p, q \) position and zeros elsewhere.

Now, for each \( j, 1 \leq j \leq k \), let

\[
K_{pj} = \begin{bmatrix}
\lambda_j & 1 & 0 & \cdots & 0 \\
\lambda_j + h_j & 1 & \ddots & \\
& \ddots & \ddots & 1 \\
0 & \cdots & & \lambda_j + (p_j - 1)h_j
\end{bmatrix}
\]

then for all \( h_j \neq 0 \) sufficiently small, \( F \) is \( H \)-analytic at \( K = K_{p_1} + \cdots + K_{p_k} \) (since \( F \) is \( H \)-analytic in a neighborhood of \( C \)).

Let \( Q_j = (q(j)_{rs}) \), \( r, s = 1, \cdots, p_j \), where \( q(j)_{rs} = 0 \) for \( r > s \) and \( q(j)_{rs} = (-1)^{s(r-s)} / (s-r)! h_j^{s-r} \) for \( r \leq s \), then \( Q_j^{-1} = (\tilde{q}(j)_{rs}) \) where \( \tilde{q}(j)_{rs} = 0 \) for \( r > s \) and \( \tilde{q}(j)_{rs} = 1 / (s-r)! h_j^{s-r} \) for \( r \leq s \); also

\[
Q_j K_{pj} Q_j^{-1} = D_{pj} = \text{diag}(\lambda_j + (i - 1)h_j), \quad i = 1, \cdots, p_j.
\]

Now, let \( Q = Q_1 + \cdots + Q_k \), then \( Q K Q^{-1} = \Lambda = D_{p_1} + \cdots + D_{p_k} \), the canonical form of \( K \). By (i), \( F \) is \( H \)-analytic at \( \Lambda \), and as in (3.4),

\[
F(\Lambda) = \sum_{i=1}^{n} F(\Lambda)_{ii} E_{ii}.
\]

By (i), \( F(K) = Q^{-1} F(\Lambda) Q \), therefore, for \( 0 \leq i \leq p_j - 1 \), \( F(K)_{r_j+r_s+i} = \sum_{s=0}^{i} q(j)_{r_j+r_s+i} F(\Lambda)_{r_j+r_s+i} q(j)_{r_s+i} \). Thus, by the first part of this proof and the definitions of \( q(j)_{rs} \) and \( \tilde{q}(j)_{rs} \),

\[
F(K)_{r_j+r_s+i} = \frac{1}{h_j^i} \sum_{s=0}^{i} \frac{(-1)^{s+i} g(\lambda_j + sh_j)}{s! (i - s)!}
\]

\[
= \frac{1}{i! h_j^i} \sum_{s=0}^{i} (-1)^{s+i} \binom{i}{s} g(\lambda_j + sh_j) = \frac{\Delta^i g(\lambda_j)}{i! h_j^i}.
\]

Since \( \lim_{h_j \to 0} \Delta^i g(\lambda_j) / h_j^i = g^{(i)}(\lambda_j) \) [6],

\[
\lim_{\sum |h_j| \to 0} K = C,
\]
and the $F_{rs}$ are analytic and therefore continuous in a neighborhood of the components of $C$, it follows that

$$F(C)_{rjrj+i} = \lim_{h_j \to 0} F(K)_{rjrj+i} = g(i_0(\lambda_j))/i!.$$  

Thus (3.3) is proven and hence Theorem 3.1.

It might here be noted that (i) alone is not sufficient for $F(Z)$ to be a primary matric function, as is shown by the function $F(Z) = \sum_{i=1}^{n} F_{ii}E_{ii}$, where $F_{ii} = \sum_{k=1}^{n} z_{kk} = \text{tr}(Z)$. The component functions $F_{ij}$ are analytic functions of the $z_{rs}$ of $Z$ and therefore $F$ is $H$-analytic; also, for $Y = P^{-1}ZP$, $F(Y) = P^{-1}F(Z)P$. However $F_{ii}$ is not a function of only $z_{ii}$ when $Z$ is a diagonal (or upper triangular) matrix which is necessary for a primary matric function.

It might be further noted, since $F(X)$ is diagonal when $X$ is diagonal, that if $X$ is restricted to the algebra $\mathcal{D}$ of $n \times n$ diagonal matrices, then $F(X)$ is also a function on $\mathcal{D}$. Ringleb [5] gave a necessary and sufficient condition for a function to be $H$-analytic in an algebra; namely, the (analytic) component functions must satisfy a certain set of linear homogeneous partial differential equations of the first order with constant coefficients which depend only on the structure of the algebra. For the algebra $\mathcal{D}$, this necessary and sufficient condition for a function $F(T) = \sum_{i=1}^{n} F(T)_{ii}E_{ii}$ to be $H$-analytic in $\mathcal{D}$ at a matrix $T = \text{diag}(t_{jj})$ is

$$\frac{\partial F(T)_{ii}}{\partial t_{jj}} = 0$$  

for $i \neq j$.

Thus hypothesis (ii) of Theorem 3.1 could be restated as follows: Let $F$ also be such that, when restricted to the algebra $\mathcal{D}$, $F$ is $H$-analytic in $\mathcal{D}$ at any diagonal matrix in $\mathcal{D}$.

References