A SUFFICIENT CONDITION FOR A MATRIC FUNCTION TO BE A PRIMARY MATRIC FUNCTION

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1. Introduction. A primary matric function is defined to be a matric function (that is, a mapping whose range and domain are sets of \( n \times n \) matrices) arising from a scalar function of a complex variable. It has been shown [1] that primary matric functions are \( H \)-analytic. In this paper other necessary conditions for a primary matric function will be exhibited and it will then be shown that these conditions are also sufficient for a matric function to be a primary function.

We will first use a form of the definition of a primary function proposed by Frobenius and later use an equivalent form proposed by Giorgi [4]. Frobenius proposed that if the scalar function \( f(z) \) is analytic at the eigenvalues of \( Z \) in \( \mathfrak{M} \) (the algebra of square matrices of order \( n \) over the complex field) then \( f(Z) \) shall be defined by

\[
    f(Z) = \frac{1}{2\pi i} \int_C \frac{f(\lambda)}{\lambda I - Z} \, d\lambda,
\]

where \( C \) is a set of admissible closed paths enclosing each of the distinct eigenvalues of \( Z \). That is, the components of \( f(Z) \) are the integrals over \( C \) of the corresponding components of the matrix \( f(\lambda)(\lambda I - Z)^{-1} \)./\( 2\pi i \).

We wish to exhibit sufficient conditions on a matric function \( F(Z) \) such that there will exist a scalar function \( g(z) \) for which \( F(Z) = g(Z) \) where \( g(Z) \) may be computed as in (1.1).

2. Necessary conditions. It has previously been shown in [1] that primary matric functions are \( H \)-analytic in \( \mathfrak{M} \), that is, the component functions of a primary function \( g(Z) \) are analytic functions of the components \( z_{ij} \) of \( Z \), for \( Z \) in an \( \mathfrak{M} \)-neighborhood of a matrix at which \( g(Z) \) is defined.

If \( g(z) \) is a scalar function defined at a matrix \( X \), that is, \( g(z) \) is analytic at the eigenvalues of \( X \), and if \( Y \) is such that for some non-singular matrix \( P \), \( Y = P^{-1}XP \), then \( g \) is defined at \( Y \) and \( g(Y) = P^{-1}g(X)P \), as can be seen from (1.1).

If \( Z \) is a matrix whose eigenvalues lie in the domain of analyticity of \( g(z) \), then the \( r, s \) component of \( g(Z) \) is given by

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\[ g(Z)_{rs} = \frac{1}{2\pi i} \int_{c} g(\lambda)(\lambda I - Z)_{rs}^{-1} d\lambda, \]

where \((\lambda I - Z)^{-1}\) is the \(r, s\) component of \((\lambda I - Z)^{-1}\). For an upper triangular matrix \(Z = (z_{ij})\), \(z_{ij} = 0\) for \(i > j\), a simple computation shows that \((\lambda I - Z)^{-1}\) and thus \(g(Z)_{rs}\) depend only on the \(z_{ij}\) for which \(r \leq i \leq j \leq s\) and is zero for \(r > s\). In particular, \(g(Z)_{rr} = g(r_{rr})\) for \(Z\) a diagonal (or upper triangular) matrix.

3. Sufficient conditions. We shall now show that these necessary conditions are also sufficient. For convenience the norm of a matrix \(Z = (z_{ij})\) shall be defined by \(\text{norm}(Z) = \max_{i, j} |z_{ij}|\).

**Theorem 3.1.** Let \(D\) be an open domain of \(H\)-analyticity of a matric function \(F\) on \(\mathbb{M}\).

(i) Let \(F\) be such that \(X\) in \(D\) and \(Y = P^{-1}XP\) implies that \(Y\) is in \(D\) and \(F(Y) = P^{-1}F(X)P\).

(ii) Let \(F\) also be such that if \(F = (t_{ij})\), in \(D\), is a diagonal matrix, then \(F(T)_{rr}\) is a function of only \(t_{rr}\), where \(F(T)_{rr}\) is the \(r, r\) component of \(F(T)\), that is

\[ F(T)_{rr} = g_{rr}(t_{rr}). \]

Then there exists a scalar function \(g(z)\) such that for all \(Z\) in \(D\), \(g(Z) = F(Z)\).

**Proof.** Let \(C\) be a Jordan form for a matrix \(Z\) at which \(F\) is \(H\)-analytic, then \(C\) is a direct sum \(C_{p_1} + \cdots + C_{p_k}\) of canonical blocks of the form

\[ C_{p_i} = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ \lambda_i & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & \lambda_i & 1 \end{bmatrix} \]

with \(p_i\) rows and columns. (The \(\lambda_i\) occurring in different \(C_{p_i}\) need not be distinct.)

From (i) and Lemma 4.1 of [2] it follows that \(F(C)\) commutes with all matrices that commute with the canonical matrix \(C\). It is known that a matrix \(F(C)\) satisfying this condition must be a direct sum \(P_1(C_{p_1}) + \cdots + P_k(C_{p_k})\), where
and \( \alpha_{im} = \alpha_{jm} \) for \( \lambda_i = \lambda_j \) (see Turnbull and Aitken [7]).

Now, using a definition proposed by G. Giorgi which is equivalent to (1.1) [4] for \( g(Z) \) where \( g(z) \) is a scalar function, it is seen that the theorem will be proven if there exists a scalar function \( g(z) \) such that, for \( C = P^{-1}ZP \), where \( Z \) is any matrix at which \( F \) is \( H \)-analytic,

\[
(3.2) \quad \alpha_{im} = g^{(m-1)}(\lambda_j)/(m - 1)!
\]

or,

\[
(3.3) \quad F(C)_{r_jr_{j+i}} = g^{(i)}(\lambda_j)/i!, \quad j = 1, \ldots, k, i = 0, \ldots, p_j - 1,
\]

where \( F(C)_{r_jr_{j+i}} \) is the \( r_j, r_j+i \) component of \( F(C) \) (that is, \( F_{r_jr_{j+i}} \) evaluated at the components of \( C \) and \( r_j = 1 + \sum_{i=1}^{k} p_i \) for \( 1 < j \leq k \), \( r_j = 1 \) for \( j = 1 \). (This choice of \( r_j \) proves (3.2) for components in the first row of each triangular block \( P_j(C_{pj}) \) of \( F(C) \) associated with \( C_{pj} \) of \( C \), which is all that is necessary, since in any such block, the values on any super diagonal are all equal.)

We shall first exhibit a scalar function \( g(z) \) which is determined by \( F \) and then show that this function has the required property (3.3).

Let \( Z \) be an arbitrary but fixed matrix such that \( F \) is \( H \)-analytic in a neighborhood of \( Z \); then \( Z \) is similar to an upper triangular matrix \( X = (x_{ij}) \) whose eigenvalues are the \( x_{ii} \). By (i) \( F \) is \( H \)-analytic in a neighborhood of \( X \). Choose any matrix \( Y = (y_{ij}) \) such that \( y_{ij} = x_{ij} \) and \( y_{ii} \neq y_{jj} \) for \( i \neq j \), and \( |y_{ii} - x_{ii}| < \epsilon \), where \( \epsilon \) is sufficiently small such that \( F \) is \( H \)-analytic at \( Y \) (such an \( \epsilon \) exists since \( F \) is \( H \)-analytic in a neighborhood of \( X \)). \( Y \) is similar to a diagonal matrix \( A = \text{diag}(y_{ii}) \) with distinct eigenvalues \( y_{ii} \) and by (i) \( F \) is \( H \)-analytic at \( A \). Now, \( A \) is similar to a diagonal matrix \( B \) obtained from \( A \) by permuting, say \( y_{ii} \) and \( y_{jj} \), and by (i), this same permutation is performed on \( F(A) \) in order to obtain \( F(B) \). Thus by (ii), \( g_{ii}(y_{ii}) = F(B)_{ii} = F(A)_{jj} = g_{jj}(y_{jj}) \). Hence for any \( j \), \( g_{jj}(z) = g_{ii}(z) \) for \( i = 1, \ldots, n \) and \( |z - x_{jj}| < \epsilon \) and therefore, since the \( F_i \) are analytic, there exists a function \( g(z) = g_{ii}(z) (= g_{ii}(z), i = 2, \ldots, n) \), analytic in the open circular domains \( |z - x_{jj}| < \epsilon, j = 1, \ldots, n \), where the \( x_{jj} \) are the eigenvalues of \( Z \). Thus, since \( Z \) is an arbitrary matrix in \( D \), there exists a function \( g(z) \) which is analytic at the eigenvalues of all matrices in \( D \).
In order to show that \( g(z) \) satisfies (3.3) we first note, from (3.1), that if \( F \) is \( H \)-analytic in a neighborhood of a canonical matrix \( C \), then \( F(C) \) may be written

\[
F(C) = \sum_{i=1}^{k} \sum_{s=0}^{p_i-1} \sum_{t=0}^{p_i-s-1} F(C)_{r_i+r_i+t} E_{r_i+s, r_i+s+t}
\]

where \( r_i = 1 + \sum_{i=1}^{t-1} p_i \) and \( E_{pq} \) is the matrix with a 1 in the \( p, q \) position and zeros elsewhere.

Now, for each \( j, 1 \leq j \leq k \), let

\[
K_{pj} = \begin{pmatrix}
\lambda_j & 1 & 0 & \cdots & 0 \\
\lambda_j + h_j & 1 & \cdot & \cdot & \cdot \\
\lambda_j + 2h_j & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \lambda_j + (p_j - 1)h_j
\end{pmatrix}
\]

then for all \( h_j \neq 0 \) sufficiently small, \( F \) is \( H \)-analytic at \( K = K_{p_1} + \cdots + K_{p_k} \) (since \( F \) is \( H \)-analytic in a neighborhood of \( C \)).

Let \( Q_j = (q(j)_{rs}) \), \( r, s = 1, \cdots, p_j \), where \( q(j)_{rs} = 0 \) for \( r > s \) and \( q(j)_{rs} = (-1)^{r+s}/(s-r)!h_j^{s-r} \) for \( r \leq s \), then \( Q_j^{-1} = (\tilde{q}(j)_{rs}) \) where \( \tilde{q}(j)_{rs} = 0 \) for \( r > s \) and \( \tilde{q}(j)_{rs} = 1/(s-r)!h_j^{s-r} \) for \( r \leq s \); also

\[
Q_j K_{pj} Q_j^{-1} = D_{pj} = \text{diag}(\lambda_j + (j - 1)h_j), \quad i = 1, \cdots, p_j.
\]

Now, let \( Q = Q_1 + \cdots + Q_k \), then \( QKQ^{-1} = \Lambda = D_{p_1} + \cdots + D_{p_k} \), the canonical form of \( K \). By (i), \( F \) is \( H \)-analytic at \( \Lambda \), and as in (3.4),

\[
F(\Lambda) = \sum_{i=1}^{n} F(\Lambda)_{ii} E_{ii}.
\]

By (i), \( F(K) = Q^{-1} F(\Lambda) Q \), therefore, for \( 0 \leq i \leq p_j - 1 \), \( F(K)_{r_j r_j+i} = \sum_{s=0}^{i} q(j)_{r_j s} F(\Lambda)_{r_j s} + q(j)_{r_j t} E_{r_j t+i} \). Thus, by the first part of this proof and the definitions of \( q(j)_{rs} \) and \( \tilde{q}(j)_{rs} \),

\[
F(K)_{r_j r_j+i} = \frac{1}{h_j^i} \sum_{s=0}^{i} \frac{(-1)^{s+t} g(\lambda_j + sh_j)}{s!(i-s)!}
\]

\[
= \frac{1}{h_j^i} \sum_{s=0}^{i} \frac{(-1)^{s+t}}{s!} \binom{i}{s} g(\lambda_j + sh_j) = \frac{\Delta^i g(\lambda_j)}{i! h_j^i}.
\]

Since \( \lim_{h_j \to 0} \Delta^i g(\lambda_j)/h_j^i = g^{(i)}(\lambda_j) \) [6],

\[
\lim_{\sum_{i=1}^{k} h_i \to 0} K = C,
\]
and the $F$, are analytic and therefore continuous in a neighborhood of the components of $C$, it follows that

$$F(C)_{rjrj+i} = \lim_{h_j \to 0} F(K)_{rjrj+i} = g^{(i)}(\lambda_j)/i!.$$ 

Thus (3.3) is proven and hence Theorem 3.1.

It might here be noted that (i) alone is not sufficient for $F(Z)$ to be a primary matric function, as is shown by the function $F(Z) = \sum_{i=1}^{n} F_{ii}E_{ii}$, where $F_{ii} = \sum_{k=1}^{n} z_{kk} = tr(Z)$. The component functions $F_{ij}$ are analytic functions of the $z_{rs}$ of $Z$ and therefore $F$ is $H$-analytic; also, for $Y = P^{-1}ZP$, $F(Y) = P^{-1}F(Z)P$. However $F_{ii}$ is not a function of only $z_{ii}$ when $Z$ is a diagonal (or upper triangular) matrix which is necessary for a primary matric function.

It might be further noted, since $F(X)$ is diagonal when $X$ is diagonal, that if $X$ is restricted to the algebra $\mathcal{D}$ of $n \times n$ diagonal matrices, then $F(X)$ is also a function on $\mathcal{D}$. Ringleb [5] gave a necessary and sufficient condition for a function to be $H$-analytic in an algebra; namely, the (analytic) component functions must satisfy a certain set of linear homogeneous partial differential equations of the first order with constant coefficients which depend only on the structure of the algebra. For the algebra $\mathcal{D}$, this necessary and sufficient condition for a function $F(T) = \sum_{i=1}^{n} F(T)_{ii}E_{ii}$ to be $H$-analytic in $\mathcal{D}$ at a matrix $T = \text{diag}(t_{ij})$ is

$$\frac{\partial F(T)_{ii}}{\partial t_{ij}} = 0 \quad \text{for } i \neq j.$$ 

Thus hypothesis (ii) of Theorem 3.1 could be restated as follows: Let $F$ also be such that, when restricted to the algebra $\mathcal{D}$, $F$ is $H$-analytic in $\mathcal{D}$ at any diagonal matrix in $\mathcal{D}$.

References


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