FUNCTIONAL EQUATIONS INVOLVING A PARAMETER¹

M. ALTMAN

1. The present note concerns the examination of nonlinear functional equations depending on a parameter. We investigate here the iterative method described in paper [1] and [2], which is a generalization of Newton's classical method. Another abstract formalism for Newton's method has been given first by L. V. Kantorovich (for references see [4]) and applied by him to the examination of operator equations in Banach spaces.

The main point here is the application of the majorant method, which was used by Kantorovich [4] and also in paper [3].

The results stated here make it possible to find an error estimation of the exact solution in the case when the solution of a suitable approximate equation is given.

An application to approximate solutions of operator equations in Hilbert space will be given in another note.

Let $X$ and $M$ be two Banach spaces, and let $F(x, \mu)$ be a nonlinear continuous functional defined on the space $X+M$, where $x$ and $\mu$ are in some closed spheres in $X$, $M$ with centres $x_0$, $\mu_0$, respectively.

Consider the nonlinear functional equation

(1) $F(x, \mu) = 0.$

Let us assume that $F(x, \mu)$ is differentiable in Fréchet's sense in the spheres mentioned above with respect to each of the two variables $x$, $\mu$ separately. Denote by

$$f(x, \mu) = F'(x, \mu) = F'_x(x, \mu)$$

the partial Fréchet derivative of $F(x, \mu)$.

Putting

$$f_n = f(x_n, \mu) = F'_x(x_n, \mu)$$

we choose a sequence of elements $y_n \in X$, $\mu \in M$ such that

(2) $\|y_n\| = 1, \quad f_n(y_n, \mu) = \|f_n\|, \quad n = 0, 1, 2, \ldots$

provided that such a choice is possible.

The iterative process for solving equation (1) is defined as in papers [1] and [2]:

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(3) \[ x_1(\mu) = x_0 - \frac{F(x_0, \mu)}{f_0(y_0, \mu)} y_0; \]
\[ x_{n+1}(\mu) = x_n(\mu) - \frac{F(x_n, \mu)}{f_n(y_n, \mu)} y_n. \]

Let us further assume that the second Fréchet derivative \( F''(x, \mu) = F''_{xx}(x, \mu) \) of \( F(x, \mu) \) exists for \( x \) in some sphere of \( X \) with centre \( x_0 \) and that the derivatives \( \partial F(x, \mu)/\partial \mu, \partial F'(x_0, \mu)/\partial \mu \) and \( \partial F''(x, \mu)/\partial \mu \) exist where \( \mu \) belongs to some sphere of \( M \) with centre \( \mu_0 \).

Consider now the real equation

(4) \[ Q(z, \nu) = 0, \]

where \( Q(z, \nu) \) is a real function of the real variables \( z, \nu \), being twice continuously differentiable in the intervals \((z_0, z')\) and \((\nu_0, \nu')\). Put \( Q'(z, \nu) = Q'(z, \nu) \) and \( Q''(z, \nu) = Q''(z, \nu) \).

Following the argument of paper [3] let us say that equation (1) possesses a real majorant equation (4), if the following conditions are satisfied:

(1°) \[ Q'(z_0, \nu) \neq 0 \quad \text{and} \quad B = -\frac{1}{Q'(z_0, \nu)} > 0; \]

(2°) \[ \|F(x_0, \mu)\| \leq Q(z_0, \nu); \]

(3°) \[ \frac{1}{\|F'(x_0, \mu)\|} \leq B; \]

(4°) \[ \|F''(x, \mu)\| \leq Q''(z, \nu) \quad \text{if} \quad \|x - x_0\| \leq z - z_0 \leq z' - z_0, \]

provided that \( \mu \) and \( \nu \) are fixed.

The following theorem of paper [3] will be used in the sequel:

**Theorem (a).** If for fixed \( \mu \) and \( \nu \) equation (1) possesses a real majorant equation (4), and if equation (4) has a real root \( z^* \) in the segment \( (z_0, z') \), then equation (1) has a solution \( x^* \), where \( \|x^* - x_0\| \leq z' - z_0 \), and the sequence of approximate solutions \( x_n \) constructed by process (3) converges to it. Moreover, we have the estimate

(5) \[ \|x_n - x^*\| \leq z^* - z_n, \]

where \( z_n \) is defined by Newton's classical process, i.e.

(6) \[ z_{n+1}(\nu) = z_n(\nu) - \frac{Q(z_n, \nu)}{Q'(z_n, \nu)}. \]
Suppose now that the approximate solution $x_0$ of equation (1) is given for a certain value $\mu_0$ of the parameter and we are interested in the solution of this equation for some other value $\mu$ of the parameter. The following theorem concerns this case.

**Theorem 1.** Let us assume that the following conditions are satisfied:

(1°) $Q'_1(z_0, \nu_0) \neq 0$ and $B = -\frac{1}{Q'(z_0, \nu_0)} > 0$;

(2°) $| F(x_0, \mu_0) | \leq Q(z_0, \nu_0);$

(3°) $\frac{1}{\| F'(x_0, \mu_0) \|} \leq B;$

(4°) $\| F''(x, \mu_0) \| \leq Q''(z, \nu_0)$ if $\| x - x_0 \| \leq z - z_0 \leq z' - z_0$;

(5°) $\left| \frac{\partial}{\partial \mu} F(x_0, \mu) \right| \leq \frac{\partial}{\partial \nu} Q(z_0, \nu)$ if $\| \mu - \mu_0 \| \leq \nu - \nu_0 \leq \nu' - \nu_0$;

(6°) $\left| \frac{\partial}{\partial \mu} F'(x_0, \mu) \right| \leq \frac{\partial}{\partial \nu} Q'(z_0, \nu)$ if $\| \mu - \mu_0 \| \leq \nu - \nu_0 \leq \nu' - \nu_0$;

(7°) $\left| \frac{\partial}{\partial \mu} F''(x, \mu) \right| \leq \frac{\partial}{\partial \nu} Q''(z, \nu)$ if $\| \mu - \mu_0 \| \leq \nu - \nu_0 \leq \nu' - \nu_0$

and $\| x - x_0 \| \leq z - z_0 \leq z' - z_0$.

If equation (4) possesses a real solution $z(\nu)$, $(z_0 \leq z(\nu) \leq z')$, for some $\nu$, $(\nu_0 \leq \nu \leq \nu')$, then equation (1) has a solution $x(\mu)$ if $\| \mu - \mu_0 \| \leq \nu - \nu_0 \leq \nu' - \nu_0$ and the sequence of approximate solutions $x_n(\mu)$ defined by process (3) converges to it. Moreover, we have

$$\| x(\mu) - x_0 \| \leq z(\nu) - z_0.$$

**Proof.** In order to prove this theorem it is sufficient to show that the conditions of Theorem (a) are satisfied. First of all we shall show that condition (2°) of the preceding theorem is fulfilled. In fact, we have by (5°)

$$| F(x_0, \mu) | = \left| F(x_0, \mu_0) + \int_{\mu_0}^{\mu} \frac{\partial}{\partial \mu} F(x_0, \mu) d\mu \right| \leq Q(z_0, \nu_0) + \int_{\nu_0}^{\nu'} \frac{\partial}{\partial \nu} Q(z_0, \nu) d\nu = Q(z_0, \nu_0) + Q(z_0, z) - Q(z_0, \nu_0) = Q(z_0, \nu).$$

Further, we get by (6°), (1°) and (3°)
\[ \| F'(x_0, \mu) \| = \| F'(x_0, \mu_0) + \int_{\mu_0}^{\mu} \frac{\partial}{\partial \mu} F'(x_0, \mu) d\mu \| \]
\[ \geq \| F'(x_0, \mu_0) \| - \left( \int_{\mu_0}^{\mu} \frac{\partial}{\partial \mu} F'(x_0, \mu) d\mu \right)^2 \]
\[ \geq \| F'(x_0, \mu_0) \| \left( 1 - \frac{1}{\| F'(x_0, \mu_0) \|} \| \int_{\mu_0}^{\mu} \frac{\partial}{\partial \mu} F'(x_0, \mu) d\mu \| \right) \]
\[ \geq \| F'(x_0, \mu_0) \| \left( 1 - \frac{\int_{\nu_0}^{\nu} Q'(x_0, \nu) d\nu}{\| F'(x_0, \mu_0) \|} \right) \]
\[ \geq \| F'(x_0, \mu_0) \| \left( 1 + \frac{Q'(z_0, \nu) - Q'(z_0, \nu_0)}{Q'(z_0, \nu_0)} \right) \]
\[ = \frac{\| F'(x_0, \mu_0) \|}{Q'(z_0, \nu_0)} Q'(z_0, \nu) \geq -Q'(z_0, \nu), \]

if the last expression is positive.

We have now to prove that \( Q'(z_0, \nu) \) is negative. For this purpose we shall show that \( Q''(z, \nu) \) is non-negative. We have by (7°)
\[ \| F''(x, \mu) \| \leq \| F''(x, \mu_0) \| + \left( \int_{\mu_0}^{\mu} \frac{\partial}{\partial \mu} F''(x, \mu) d\mu \right) \]
\[ \leq Q''(z, \nu_0) + \int_{\nu_0}^{\nu} Q''(x, \nu) d\nu \]
\[ = Q''(z, \nu_0) + Q''(z, \nu) - Q''(z, \nu_0) \]
\[ = Q''(z, \nu). \]

If \( Q'(z_0, \nu) \) were non-negative we should have \( Q'(z, \nu) \geq 0 \) since \( Q''(z, \nu) \geq 0 \). Hence we get by (1°) and (5°) \( Q(z, \nu) \geq Q(z_0, \nu) \geq Q(z_0, \nu_0) > 0 \). But this leads to a contradiction because equation (4) has a real solution. Thus, we conclude that condition (1°) is satisfied. It remains to prove that condition (4°) of Theorem (a) is also satisfied, i.e. \( \| F'(x, \mu) \| \leq Q'(z, \nu) \) if \( \| x - x_0 \| \leq z - z_0 \leq z' - z_0 \), and \( \| \mu - \mu_0 \| \leq \nu - \nu_0 \leq \nu' - \nu_0 \). But this verification has already been obtained above, and thus the theorem is proved.

**Remark 1.** The error estimate is given by the formula
\[ \| x_n(\mu) - x(\mu) \| \leq z(\nu) - z_n(\nu). \]

This remark follows from (5).

**Remark 2.** Condition (2°) can be replaced by condition
This remarks follows from the proof of Theorem (a).

Consider now the following particular case of a functional equation depending on a parameter:

\[(7) \quad F(x, \mu) = G(x) + \mu H(x) = 0,\]

where \(G(x)\) and \(H(x)\) are nonlinear, continuous functionals on \(X\) and \(\mu\) is a real number. Suppose that a solution of equation (7) is given for \(\mu_0 = 0\). Applying Theorem 1 we obtain the following

**Theorem 2.** Let us assume that \(G(x)\) and \(H(x)\) are twice continuously differentiable in the sense of Fréchet and the following conditions are fulfilled:

1. \(G'(x_0) = 0.\)
2. \(\frac{1}{\|G'(x_0)\|} \leq B.\)
3. \(\|G''(x)\| \leq K \quad \text{if} \quad \|x - x_0\| \leq z' - z_0;\)
4. \(H(x_0) \leq \eta;\)
5. \(\|H'(x_0)\| \leq \alpha;\)
6. \(\|H''(x)\| \leq \beta \quad \text{if} \quad \|x - x_0\| \leq z' - z_0,\)

and

\[(7) \quad \frac{(1 - \alpha B\nu)^2}{B^2} - 2\nu(K + \nu\beta) \geq 0; \quad 0 < \alpha B\nu < 1.\]

Then equation (7) has a solution if \(|\mu| \leq \nu\) and the sequence of approximate solutions \(x_n\) defined by process (3) converges to it. Moreover, the solution \(x^*\) satisfies the inequality \(\|x^* - x_0\| \leq z(\nu)\) and conditions (5) and (6) hold, provided that the majorant equation (4) is replaced by the following one:\(^2\)

\[(8) \quad Q(z, \nu) = \frac{K + \nu\beta}{2} z^2 - \frac{1 - \alpha B\nu}{B} z + \nu\eta = 0, \quad (z_0 = 0, \nu_0 = 0).\]

**Proof.** It is easy to verify that all conditions \((1^o)-(7^o)\) of Theorem 1 are satisfied.

**Remark 3.** Instead of the majorant equation (8) we can use the following one:\(^2\)

\[^2\] It seems to be interesting to notice that these majorant equations are the same as those considered by Kantorovich [4].
In this case condition (6) should be replaced by condition (10)

\[ \| H''(x) \| \leq \beta(r) \text{ if } \| x - x_0 \| \leq r. \]

All assertions of Theorem 2 hold if equation (9) has a positive root for
\[ \| \mu \| \leq \nu. \]

**Remark 4.** Notice that Corollary 2 in [3, p. 23] may be considered
as a particular case of Theorem 2 if we put

\[ F(x, \mu) = [F(x) - F(x_0)] + \mu F(x_0), \quad (\mu_0 = 1). \]

2. In this section we are concerned with the error estimation for
the approximate solution of the functional equation

\[ F(x) = 0, \]

where \( F(x) \) is a nonlinear continuous functional defined on the Banach
space \( X \).

At the same time we consider the approximate functional equation

\[ G(x) = 0, \]

where \( G(x) \) is also a nonlinear continuous functional defined on \( X \).
Suppose that \( x_0 \) is a solution of equation (12). In order to find how
near the solution of equation (11) is to \( x_0 \) we introduce the following
functional equation depending on a parameter:

\[ F(x, \mu) = G(x) + \mu [F(x) - G(x)] = G(x) + \mu H(x) = 0. \]

Suppose that both \( F(x) \) and \( G(x) \) are twice continuously differentiable
in the sense of Fréchet. We are now in a position to apply Theorem 2.
Hence we get

**Corollary 1.** Let us assume that the following conditions are ful-
inished.

1. \( G(x_0) = 0, \]
2. \( 1/\| G'(x_0) \| \leq B, \]
3. \( \| G''(x) \| \leq K \text{ if } \| x - x_0 \| \leq z' - z_0, \]
4. \( F(x_0) \leq \eta, \]
5. \( \| F'(x_0) - G'(x_0) \| \leq \alpha, \]
6. \( \| F''(x) - G''(x) \| \leq \beta \text{ if } \| x - x_0 \| \leq z' - z_0. \]
7. \( (1 - \alpha B)^2/B^2 - 2\eta(K + \beta) \geq 0, \quad (\alpha B \leq 1). \]

Then equation (11) has a solution \( x^* \) such that

\[ \| x^* - x_0 \| \leq z_1, \]
where \( z_1 \) is the smallest root of the equation

\[
\frac{K + \beta}{2} z^2 - \frac{1 - \alpha B}{B} z + \eta = 0.
\]

This estimation may be useful especially in the case, when the expression (2) is more simple than the corresponding one for the functional \( F \). We shall now apply the estimation obtained above replacing \( G(x) \) by

\[
G(x) = F(x_0) + F'(x_0)(x - x_0).
\]

As the initial approach, which appears in Corollary 1, we take now the solution \( x_1 \) of equation

\[
G(x_1) = F(x_0) + F'(x_0)(x_1 - x_0) = 0.
\]

Condition (15), is, of course, satisfied if \( x_1 \) is defined by process (3). As a particular case of Corollary 1 we obtain

**Corollary 2.** Let us assume that the following conditions are satisfied:

1. \( |F(x_0)| \leq \eta \),
2. \( 1/\|F'(x_0)\| \leq B \),
3. \( \|F''(x)\| \leq K \) if \( \|x - x_0\| \leq z' - z_0 \),
4. \( |F(x_1)| \leq \eta_1 \),
5. \( (1 - KB^2\eta_1)^2/B^2 - 2\eta K \geq 0 \), \( (KB^2\eta_1 \leq 1) \).

Then equation \( F(x) = 0 \) has a solution \( x^* \) such that

\[
\|x^* - x_1\| \leq z_1,
\]

where \( z_1 \) is the smallest root of equation

\[
\frac{1}{2} Kz^2 - \frac{1 - B^2K\eta}{B} z - \eta_1 = 0.
\]

Let us observe that in this case the following conditions are satisfied:

1. \( G(x_1) = 0 \),
2. \( 1/\|G'(x_1)\| = 1/\|F'(x_0)\| \leq B \),
3. \( \|G''(x)\| = 0 \),
4. \( |F(x_1)| \leq \eta_1 \),
5. \( \|F'(x_1) - G'(x_1)\| = \|F'(x_1) - F'(x_0)\| \leq K\|x_1 - x_0\| \leq KB\eta = \alpha \),
6. \( \|F''(x) - G''(x)\| = \|F''(x)\| \leq K = \beta \).

But this means that all conditions (1)–(7) of Corollary 1 are satisfied provided that \( x_0 \) is replaced by \( x_1 \) and \( \alpha = KB\eta, \beta = K \).
AN UNCOUNTABLE SET OF INCOMPARABLE DEGREES

J. R. SHOENFIELD

The purpose of this note is to prove the following:\footnote{The problem solved in this paper was suggested to the author by C. Spector.}

\textbf{Theorem.} There is an uncountable set of pairwise incomparable degrees of recursive unsolvability.

By Zorn's lemma, there is a maximal set of pairwise incomparable degrees of recursive unsolvability different from 0; we must show that this set is not countable. Hence our theorem follows from:

\textbf{Lemma.} If $a_0, a_1, \cdots$ is a sequence of degrees different from 0, then there is a degree $b$ which is incomparable with each $a_n$.

\textbf{Proof.}\footnote{We use the notation of [1] in the proof.} Let $\alpha_n$ be a function of degree $a_n$; we shall construct a function $\beta$ of degree $b$. As in [1], $\beta$ is constructed by defining inductively a function $\kappa$ such that $\kappa(a) = \beta(\nu(a))$ with $\nu(a) = lh(\kappa(a))$; $\kappa$ and $\nu$ must satisfy the conditions that $\kappa(a)$ is a sequence number, $\kappa(a+1)$ extends $\kappa(a)$, and $\nu(a+1) > \nu(a)$. We then have $\beta(a) = (\kappa(a+1))_a - 1$.

Let $\kappa(0) = 1$. To define $\kappa(a+1)$, let $n = (a)_1$ and $e = (a)_2$. If $a$ is even, set

$$\kappa(a + 1) = \kappa(a) \cdot p_{\nu(a)} \exp(\{e\}^{\alpha_n}(\nu(a)) + 2)$$

if $\{e\}^{\alpha_n}(\nu(a))$ is defined, and $\kappa(a+1) = \kappa(a) \cdot p_{\kappa(a)}$ otherwise. Then clearly $\beta \neq \{e\}^{\alpha_n}$ for any function $\beta$ such that $\beta(\nu(a+1)) = \kappa(a+1)$. 

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