IMMERSION OF MANIFOLDS OF NONPOSITIVE CURVATURE

BARRETT O'NEILL

In [4] Tompkins proved that a flat compact Riemannian manifold \( M \) of dimension \( n \) cannot be (isometrically) immersed in Euclidean \((2n - 1)\)-space. Chern and Kuiper conjectured in [2] that the result holds if the Riemannian (i.e. sectional) curvature \( K \) of \( M \) is never positive. An algebraic result verifying this conjecture was found by Otsuki [3]. We shall prove:

**Theorem.** Let \( M \) be a compact \( n \)-dimensional Riemannian manifold and let \( \overline{M} \) be a complete simply connected Riemannian manifold of dimension less than \( 2n \). If the Riemannian curvatures \( K \) and \( \overline{K} \) of \( M \) and \( \overline{M} \) satisfy \( K \leq \overline{K} \leq 0 \), then \( M \) cannot be immersed in \( \overline{M} \).

Simple examples involving spheres and tori show that the theorem fails if either inequality or the simple connectedness of \( \overline{M} \) is deleted.

Following [1] we express the second fundamental form information of an immersion \( i: M \to \overline{M} \) in terms of the difference transformation \( T \), a function which assigns to each vector \( x \) in \( M_m \) (the tangent space to \( M \) at \( m \in M \)) a linear transformation \( T_x \) from \( M_m \) to the orthogonal complement \( M_m^\perp \) of \( di(M_m) \) in \( M_\perp(m) \). The natural definition of \( T \) is in terms of the notion of difference of two connections [1], however it may be described in terms of the classical second fundamental form \( S \) as follows: \( S \) is a function which assigns to each vector \( z \in M_m^\perp \) a symmetric linear operator \( S_z \) on \( M_m \). If \( x \in M_m \) and \( z \in M_m^\perp \), let \( T_x(z) = S_z(x) \); then uniquely extend \( T_x \) to be a skew-symmetric operator on all of \( M_\perp(m) \). For our purposes, as indicated above, we need only the portion of \( T_x \) defined on \( di(M_m) \) or, equivalently, \( M_m \). As a function of \( x, y \in M_m \), \( T_x(y) \) is bilinear and symmetric.

The difference transformation relates the Riemannian curvatures of \( M \) and \( \overline{M} \) as follows: if \( x \) and \( y \) span a plane \( P \) in \( M_m \), then

\[
K(P) = \frac{\langle T_x(x), T_y(y) \rangle - \| T_x(y) \|^2}{\| x \wedge y \|^2} + \overline{K}(di(P)).
\]

This formula, the Gauss equation, is readily obtained from the second structural equations of \( M \) and \( \overline{M} \).

The following lemma extends a well-known Euclidean fact.

**Lemma 1.** Let \( i: M \to \overline{M} \) be an immersion of a compact Riemannian...
manifold in a complete simply connected manifold with Riemannian curvature $\bar{K} \leq 0$. Then there is a point $m \in M$ and a vector $z \in M^\perp_m$ such that $\langle T_z(x), z \rangle < 0$ for all nonzero $x \in M_m$.

Proof. Fix a point $\bar{m}$ in $\bar{M}$ and use the following notation: $m$, a point of $M$ such that $i(m)$ has maximum distance from $\bar{m}$; $\sigma$: $[0, 1] \rightarrow \bar{M}$, the unique geodesic from $m$ to $i(m)$; $z$, the velocity vector $\sigma(1)$ of $\sigma$ at $i(m)$. If $x \in di(M_m)$ there is a differentiable map $r$: $[0, 1] \times [0, 1] \rightarrow M$ such that (1) $r(\cdot, 0) = \sigma$, (2) for each $v \in [0, 1]$, $r(\cdot, v)$ is a geodesic, (3) $r(0, \cdot) = \bar{m}$ and $r(1, \cdot) \in i(M)$, (4) if $X$ is the vector field on $\sigma$ such that $X(u)$ is the velocity of $r(u, \cdot)$ at $v = 0$, then $X(1) = x$. Let $l(v)$ be the length of $r(\cdot, v)$, that is, the distance from $\bar{m}$ to $r(1, v) \in i(M)$. Obviously $l'(0) = 0$ and $l''(0) \leq 0$. The vanishing of the first variation implies $z \in M^\perp_m$. By the Synge formula for the second variation $[1]$ we have

$$sl''(0) = \int_0^1 \left\{ \|X'\|^2 - \bar{K}(\sigma, X)\|\sigma \wedge X\| \right\} + \langle T_z(x), z \rangle$$

where $s$ is the length of $\sigma$, and $X'$ is the covariant derivative of $X$. Since $\bar{K} \leq 0$, the integral term is positive if $x \neq 0$, hence $\langle T_z(x), z \rangle < 0$.

Proof of the theorem. Following Tompkins' scheme we reduce the proof to a problem in linear algebra. Let $i$: $M \rightarrow \bar{M}$ be an immersion, where $M$ and $\bar{M}$ are as described in the theorem, except that no restriction is made on the dimension of $\bar{M}$. Let $m$ and $z$ be as in Lemma 1. From the formula (1) and the condition $K \leq \bar{K}$ we get $\langle T_z(x), T_y(y) \rangle \leq \|T_z(y)\|^2$ for all $x, y \in M_m$. We need only prove that dimension $M^\perp_m \geq n$, and this follows from

**Lemma 2.** Let $U$ and $V$ be finite-dimensional real vector spaces, $V$ with an inner product. Suppose that for each $x \in U$ there is a linear transformation $T_x$: $U \rightarrow V$ such that:

1. $\langle T_x(x), T_y(y) \rangle \leq \|T_x(y)\|^2$ for all $x, y \in U$.
2. $T_x(y)$ is bilinear and symmetric in $x$ and $y$.
3. There is a vector $z \in V$ such that $\langle T_z(x), z \rangle < 0$ for all nonzero $x \in U$. Then dimension $V \geq$ dimension $U$.

Proof. Suppose the contrary; then for every $u \in U$ there is a non-zero $v \in U$ such that $T_u(v) = 0$, hence $\langle T_u(u), T_v(v) \rangle \leq 0$. Consider the real-valued function $f$ on $U - \{0\}$ for which

$$f(u) = \frac{\langle T_u(u), z \rangle}{\|T_u(u)\| \|z\|}.$$ 

Since $f$ is continuous and constant on lines through the origin, it has a
minimum, say \( f(x) \). Let \( y \) be a nonzero vector such that \( T_z(y) = 0 \) and \( \langle T_z(x), T_y(y) \rangle \leq 0 \). Note that \( T_z(x) \) and \( T_y(y) \) are independent, for otherwise we may assume \( T_z(x) + T_y(y) = 0 \), which contradicts (3).

Let \( P \) be the plane spanned by these two vectors, and let \( z^\perp \) be the set of vectors orthogonal to \( z \). Since \( T_z(x) \) is not in \( z^\perp \), the subspace \( P \cap z^\perp \) is 1-dimensional. Let \( p \) be the unique unit vector in \( P \) such that \( \langle p, z \rangle < 0 \) and \( p \) is orthogonal to \( P \cap z^\perp \). Clearly \( \langle p, z \rangle < \langle q, z \rangle \) if \( q \) is a unit vector in \( P \) different from \( p \). Now the definition of \( p \) together with the inequalities \( \langle T_z(x), T_y(y) \rangle \leq 0 \), \( \langle T_z(x), z \rangle < 0 \), \( \langle T_y(y), z \rangle < 0 \) imply that \( p \) lies between \( T_z(x) \) and \( T_y(y) \), that is, that we may write \( p = \lambda^1 T_z(x) + \mu^1 T_y(y) \). Thus \( p = T_{\lambda x + \mu y} (\lambda x + \mu y) \). In view of the minimality properties of \( p \) and \( T_z(x) \) we must have \( T_z(x) \) a positive scalar multiple of \( p \). But since \( p \) is orthogonal to \( P \cap z^\perp \), this implies \( \langle T_y(y), z \rangle \geq 0 \), a contradiction.

It is clear that the theorem holds in slightly more general form, patterned on the full Chern-Kuiper conjecture, namely: if there is at each point \( m \) of \( M \) a \( q \)-dimensional subspace \( S_m (q \geq 2) \) of \( M_m \) such that \( K \subseteq K \) holds when \( K \) is restricted to planes in any \( S_m \), then immersion is impossible if the dimension of \( M \) is less than \( n+q \).

References


Massachusetts Institute of Technology