Bibliography

1. Edwin Hewitt and H. S. Zuckerman, *The l_1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc. vol. 83 (1956) pp. 70–97.
2. André Weil, *L'intégration dans les groupes topologiques et ses applications*, Actualités Sci. Ind., no. 869, Paris, Hermann, 1940.

University of Washington and
Harvard University

HAUSDORFF INTERVAL TOPOLOGY ON A PARTIALLY ORDERED SET

YATARŌ MATSUSHIMA

We shall generalize a condition of E. S. Wolk [1] that the interval topology in a partially ordered set be Hausdorff. Let X be a partially ordered set. For each $a \in X$, let $N(a)$ be the set of all elements of X, noncomparable with a. We introduce the definition of an "*a-separating set*": any subset S of $N(a)$ such that every $x \in N(a)$ is comparable with some $y \in S$.

Theorem. *If each $a \in X$ has a finite a-separating set, then X is a Hausdorff space in its interval topology.*

Proof. Let $a \neq b$, $a, b \in X$. Let $\{a_i\}_1^r(\{b_j\}_1^s)$ be an *a-separating* (*b-separating*) *set*; we define for each of the cases the sets A and B; one checks easily in each case that A, B have the stated properties.

(1) The case where a, b are comparable.

(α) Let $a < b$. If there is an element c such that $a < c < b$, then, there exists a *c-separating set* $\{c_i\}$, so that

$$N(c) \subset \sum_{i=1}^{m} ([-\infty, c_i] + [c_i, \infty]), \quad \text{where} \quad c_i \in N(c).$$

In this case if we put

$$A = [-\infty, c] + \sum_{i=1}^{m} [-\infty, c_i], \qquad B = [c, \infty] + \sum_{i=1}^{m} [c_i, \infty],$$

Received by the editors April 17, 1959.

then A and B are both closed sets in the interval topology, and furthermore $b \overline{\in} A$, $a \overline{\in} B$ and $X = A + B$.

(β) If there is no c such that $a < c < b$, we put

$$A = [-\infty, a] + [a, \infty] \cdot \sum_{j=1}^{s} ([-\infty, b_j] + [b_j, \infty]),$$

$$B = [b, \infty] + \left(\sum_{i=1}^{r} ([-\infty, a_i] + [a_i, \infty]) \right)$$
$$\cdot \left([-\infty, b] + \sum_{j=1}^{s} ([-\infty, b_j] + [b_j, \infty]) \right).$$

Then we have $b \overline{\in} A$ since $b \overline{\in} \sum_{j=1}^{s} ([-\infty, b_j] + [b_j, \infty])$. Similarly we have $a \overline{\in} B$. Furthermore we have $X = A + B$. Indeed if $x \in X - ([-\infty, a] + [b, \infty])$, then we have either $x > a$ or $x \in N(a)$. In the first case we have $x \not< b$ by the hypothesis, then $x \in [a, \infty] \cdot N(b)$, that is, $x \in A$. In the second case $x \overline{\in} [b, \infty]$, and hence $x \in ([-\infty, b] + N(b)) \cdot N(a)$, that is, $x \in B$.

(2) The case where $b \in N(a)$.

$$A = [-\infty, a] + [a, \infty], \qquad B = \sum_{i=1}^{r} ([-\infty, a_i] + [a_i, \infty]),$$

then we have $b \overline{\in} A$, $a \overline{\in} B$; $X = A + B$.

To show that our theorem generalizes Wolk's result we prove a lemma and give a counter example.

Lemma. *If X contains no infinite diverse subsets, then each $a \in X$ has a finite a-separating set.*

Proof. For any $a \in X$ if $a_1 \in N(a)$, $N(a) \cap N(a_1) \neq 0$, then take $a_2 \in N(a) \cap N(a_1)$. If $N(a) \cap N(a_1) \cap N(a_2) = 0$, then we have

$$N(a) \subset \sum_{i=1}^{2} ([-\infty, a_i] + [a_i, \infty])$$

satisfying our requirements. If $N(a) \cap N(a_1) \cap N(a_2) \neq 0$, then we can proceed as above. However the following infinite cases do not occur by the hypothesis:

$$a_i \in N(a) \cap N(a_1) \cap \cdots \cap N(a_{i-1}), N(a) \cap N(a_1) \cap \cdots \cap N(a_i) \neq 0,$$

$i = 1, 2, \cdots$. This completes the proof.

The converse of this lemma is not true.

To show this we give an example of a lattice L_0 with infinite sets $\{a_i\}$, $\{b_j\}$, $\{c_k\}$ and $\{d_l\}$ satisfying the following conditions:

(1) $a_i > b_i > c_i > d_i$, all i;

(2) $a_i > a_{i+1} > b_i$, $c_i > d_{i+1} > d_i$, all i;

(3) $\{b_j\}$ and $\{c_k\}$ consist of pairwise noncomparable elements respectively, all j, k.

Then L_0 contains infinite diverse sets $\{b_j\}$ and $\{c_k\}$, but any element of L_0 satisfies the assumption of Theorem 1. Indeed for instance as regards $N(b_n)$ we have

$$N(b_n) \subset [-\infty, a_{n+2}] + \sum_{i=1}^{n-1} [-\infty, b_i],$$

where a_{n+2} and b_i are contained in $N(b_n)$. Similarly we can show the other cases.

COROLLARY. *If a partially ordered set X contains no infinite diverse set, then X is a Hausdorff space in its interval topology* (*E. S. Wolk*, [1]).

REMARK. One sees easily that Northam's condition (c) [2, Proposition 7], is equivalent to: every x has a finite *x-separating set.*

REFERENCES

1. E. S. Wolk, *Order-compatible topology on a partially ordered set*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 524–529.

2. E. S. Northam, *The interval topology of a lattice*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 824–827.

GUNMA UNIVERSITY, MAEBASHI, JAPAN