Bibliography


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HAUSDORFF INTERVAL TOPOLOGY ON A PARTIALLY ORDERED SET

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We shall generalize a condition of E. S. Wolk [1] that the interval topology in a partially ordered set be Hausdorff. Let $X$ be a partially ordered set. For each $a \in X$, let $N(a)$ be the set of all elements of $X$, noncomparable with $a$. We introduce the definition of an "$a$-separating set": any subset $S$ of $N(a)$ such that every $x \in N(a)$ is comparable with some $y \in S$.

Theorem. If each $a \in X$ has a finite $a$-separating set, then $X$ is a Hausdorff space in its interval topology.

Proof. Let $a \neq b$, $a, b \in X$. Let $\{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n$ be an $a$-separating ($b$-separating) set; we define for each of the cases the sets $A$ and $B$; one checks easily in each case that $A, B$ have the stated properties.

(1) The case where $a, b$ are comparable.

(a) Let $a < b$. If there is an element $c$ such that $a < c < b$, then, there exists a $c$-separating set $\{c_i\}$, so that

$$N(c) \subseteq \sum_{i=1}^m \left(]-\infty, c_i] + [c_i, \infty]\right), \quad \text{where} \quad c_i \in N(c).$$

In this case if we put

$$A = [-\infty, c] + \sum_{i=1}^m [-\infty, c_i], \quad B = [c, \infty] + \sum_{i=1}^m [c_i, \infty],$$

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then \( A \) and \( B \) are both closed sets in the interval topology, and furthermore \( b \in A, a \in B \) and \( X = A + B \).

(\( \beta \)) If there is no \( c \) such that \( a < c < b \), we put

\[
A = [-\infty, a] + [a, \infty] \cdot \sum_{j=1}^{s} ([\infty, b_j] + [b_j, \infty]),
\]

\[
B = [b, \infty] + \left( \sum_{i=1}^{r} ([\infty, a_i] + [a_i, \infty]) \right) \cdot \left( [-\infty, b] + \sum_{j=1}^{s} ([\infty, b_j] + [b_j, \infty]) \right).
\]

Then we have \( b \in A \) since \( b \in \sum_{j=1}^{s} ([\infty, b_j] + [b_j, \infty]) \). Similarly we have \( a \in B \). Furthermore we have \( X = A + B \). Indeed if \( x \in X - ([\infty, a] + [a, \infty]), \) then we have either \( x > a \) or \( x \in N(a) \). In the first case we have \( x < b \) by the hypothesis, then \( x \in [a, \infty] \cdot N(b) \), that is, \( x \in A \). In the second case \( x \in [b, \infty] \), and hence \( x \in ([\infty, b] + N(b)) \cdot N(a) \), that is, \( x \in B \).

(2) The case where \( b \in N(a) \).

\[
A = [-\infty, a] + [a, \infty], \quad B = \sum_{i=1}^{r} ([\infty, a_i] + [a_i, \infty]),
\]

then we have \( b \in A, a \in B \); \( X = A + B \).

To show that our theorem generalizes Wolk’s result we prove a lemma and give a counter example.

**Lemma.** If \( X \) contains no infinite diverse subsets, then each \( a \in X \) has a finite \( a \)-separating set.

**Proof.** For any \( a \in X \) if \( a_1 \in N(a) \), \( N(a) \cap N(a_1) \neq 0 \), then take \( a_2 \in N(a) \cap N(a_1) \). If \( N(a) \cap N(a_1) \cap N(a_2) = 0 \), then we have

\[
N(a) \subset \sum_{i=1}^{3} ([\infty, a_i] + [a_i, \infty])
\]

satisfying our requirements. If \( N(a) \cap N(a_1) \cap N(a_2) \neq 0 \), then we can proceed as above. However the following infinite cases do not occur by the hypothesis:

\[a_i \in N(a) \cap N(a_1) \cap \cdots \cap N(a_{i-1}), N(a) \cap N(a_1) \cap \cdots \cap N(a_i) \neq 0, \quad i=1, 2, \ldots .\]

This completes the proof.

The converse of this lemma is not true.

To show this we give an example of a lattice \( L_0 \) with infinite sets \( \{a_i\}, \{b_j\}, \{c_k\} \) and \( \{d_i\} \) satisfying the following conditions:
(1) $a_i > b_i > c_i > d_i$, all $i$;
(2) $a_i > a_{i+1} > b_i > c_i > d_{i+1} > d_i$, all $i$;
(3) $\{b_j\}$ and $\{c_k\}$ consist of pairwise noncomparable elements respectively, all $j$, $k$.

Then $L_0$ contains infinite diverse sets $\{b_j\}$ and $\{c_k\}$, but any element of $L_0$ satisfies the assumption of Theorem 1. Indeed for instance as regards $N(b_n)$ we have

$$N(b_n) \subset \left[ -\infty, a_{n+2} \right] + \sum_{i=1}^{n-1} \left[ -\infty, b_i \right],$$

where $a_{n+2}$ and $b_i$ are contained in $N(b_n)$. Similarly we can show the other cases.

**Corollary.** If a partially ordered set $X$ contains no infinite diverse set, then $X$ is a Hausdorff space in its interval topology (E. S. Wolk, [1]).

**Remark.** One sees easily that Northam's condition (c) [2, Proposition 7], is equivalent to: every $x$ has a finite $x$-separating set.

**References**


**Gunma University, Maebashi, Japan**