PRIME RINGS SATISFYING A POLYNOMIAL IDENTITY

EDWARD C. POSNER

THEOREM. *R is a prime ring satisfying a polynomial identity if and only if R is a subring of the ring of all $r\times r$ matrices, for some r, over a division ring D finite dimensional over its center, and R has a two-sided ring of quotients which is all of the matrix ring. (For this result when R has no zero divisors, see* [1]).

PROOF. Sufficiency is easy and omitted. We recall that B is a two-sided quotient ring of its subring A if every element of B can be written ab^{-1}, $a, b\in A$, and also $c^{-1}d$, $c, d\in A$, and if every element of B not a left (right) zero divisor has a right (left) inverse in B. B is unique given A if it exists.

To prove necessity we invoke Goldie's Theorem (2): Let R be a prime ring satisfying ($1l$), ($1r$), ($2l$), ($2r$). Then R has an $m\times m$ matrix ring over a division ring as its full ring of quotients. Here ($1l$) is: every direct sum of nonzero left ideals of R has a finite number of terms. ($2l$) is: the ascending chain condition holds for the annihilator left ideals of R. ($1r$) and ($2r$) are analogous.

We will prove that if R is a prime ring satisfying a polynomial identity, then R satisfies ($1l$) and ($2l$). The conditions for right ideals will follow similarly. Let $\sum a_\pi x_{\pi(1)} \cdots x_{\pi(n)}=0$, $a_\pi\in C$, the centroid of R, π a permutation of $1, 2, \cdots, n$, be a homogeneous multilinear identity for R. We will show that the length of a direct sum of nonzero left ideals is at most $n-1$. First let I_j, $1\leqq j\leqq n$, be left ideals invariant under the centroid of R, and let $I_1\oplus \cdots \oplus I_n$ be direct. Let $x_j\in I_j$. Then all terms in the identity whose rightmost factor is x_1, say, must add up to zero by directness and the fact that the I_j are invariant under the centroid. Here x_1 was so numbered that at least one nonzero coefficient occurs. Since R is prime, I_1 has no left annihilator, that is, we can now cancel x_1 from this identity. Continuing in this fashion, renumbering if necessary, we find $I_n=0$. If I_j are not invariant under the centroid C, the CI_j are, and $CI_1\oplus \cdots \oplus CI_n$ is still direct. For if $\sum_{j=1}^n c_j i_j=0$, $c_j\in C$, $i_j\in I_j$, $1\leqq j\leqq n$, then $\forall r\in R$, $\sum_{j=1}^n (rc_j)i_j=0$. But $rc_j\cdot i_j\in I_j$, $1\leqq j\leqq n$, so each $rc_j i_j=0$. Since R has no absolute right divisors of zero, $c_j i_j=0$, $1\leqq j\leqq n$.

To prove that R satisfies the ascending chain condition for left annihilator ideals, suppose $I_1\subsetneq I_2\subsetneq \cdots \subsetneq I_n$ are left annihilator

Received by the editors April 25, 1959.

ideals where we may suppose I_j is the total left annihilator of a right ideal K_j and $K_j \neq K_{j-1}$, $2 \leqq j \leqq n$. Let k ($\leqq n$) be the first integer such that $\exists \{\beta_\pi\} \in C$ not all zero with $\sum \beta_\pi i_{\pi(1)} \cdots i_{\pi(k)} = 0$ whenever $i_j \in I_j$, $1 \leqq j \leqq k$, the sum extended over all permutations of $1, 2, \cdots, k$, and such that $\beta_\pi \neq 0$ when π is the identity permutation. Multiply this identity on the right by K_{k-1}. $\sum' \beta_\pi i_{\pi(1)} \cdots i_{\pi(k-1)} i_{\pi(k)} K_{k-1} = 0$ where $\sum'$ is taken over those π with $\pi(k) = k$. In other words $\sum \beta_\pi i_{\pi(1)} \cdots i_{\pi(k-1)} i_k K_{k-1} = 0$, the sum extended over all permutations of $1, 2, \cdots, k-1$, $\forall i_k \in I_k$. Or $\sum \beta_\pi i_{\pi(1)} \cdots i_{\pi(k-1)} I_k K_{k-1} = 0$, $\forall i_j \in I_j$, $1 \leqq j \leqq k-1$. Now $I_k K_{k-1}$ is a two-sided ideal, and, by assumption, not zero. So by primeness, $\sum \beta_\pi i_{\pi(1)} \cdots i_{\pi(k-1)} = 0$, which contradicts the minimality of k. So the assumption that $I_k I_{k-1} \neq 0$ must be retracted, $I_{k-1} = I_k$ and not $I_{k-1} \not\subseteq I_k$.

To prove that the quotient ring Q of R is a matrix ring over a finite-dimensional division ring, it suffices to prove that Q satisfies a polynomial identity. (Actually Q satisfies the same identity as we shall see.) R satisfies a standard identity $\sum \operatorname{sgn} \pi x_{\pi(1)} \cdots x_{\pi(2p)} = 0$ where $\operatorname{sgn} \pi$ is $+1$ or -1 according as π is even or odd; in fact R, having no nilpotent ideals, is a subring of a direct sum of $p \times p$ matrix rings over fields [**3**, p. 227, Theorem 2]. We wish to prove $\sum \operatorname{sgn} \pi x_{\pi(1)} d_{\pi(1)}^{-1} \cdots x_{\pi(2p)} d_{\pi(2p)}^{-1} = 0$ for all x_i, $d_i \in R$, d_i regular in R, $1 \leqq i \leqq 2p$. But resorting to the very definition of rings of quotients [**4**, p. 118], we can write the condition that the standard identity of degree $2p$ be satisfied in a form not involving inverses at all by resorting to the definition of addition and multiplication in Q. The condition is of the form that if a certain (large) set of auxiliary elements of R satisfy one set of equations involving the x_i and d_i, and not their inverses, (namely, the set of equations which describes when uv^{-1}, u, $v \in R$ involving the x_i, d_i and auxiliary elements, is $\sum \operatorname{sgn} \pi x_{\pi(1)} d_{\pi(1)}^{-1} \cdots x_{\pi(2p)} d_{\pi(2p)}^{-1}$ in Q), then they satisfy another equation, namely, $u = 0$. For $p \times p$ matrices over fields, we know that whenever the first set of equations is satisfied by auxiliary elements, the second equation is satisfied, since $p \times p$ matrices satisfy the standard identity of degree $2p$. That is, the second equation is satisfied if the d_i were invertible in the matrix ring. (Our d_i in R may not be invertible in each matrix summand in which R is embedded, where invertible means invertible as a matrix and not as an element of R.) We wish to show that the second equation is satisfied even if the d_i are not invertible matrices. Let the coefficients in the $2p$ $p \times p$ matrices d_i be $2p$ independent transcendentals $\{y_\gamma\}$. In particular, each d_i is invertible. We may keep the x_i fixed at their given values. To say that $u = 0$ is satisfied whenever the other set of equations is satisfied says that the set of zeros of the p^2 polynomials $\{f_\alpha\}$ obtained from $u=0$ in the indeterminates $\{u_\delta\}$ (corresponding to the coefficients of the matrices of each auxiliary variable) contains the set of zeros of the other set of polynomials $\{g_\beta\}$ (obtained from the set of equations defining u). By Hilbert's *Nullstellensatz*, for some integer q which we may take to be the same for every α, $f_\alpha^q = \sum r_{\beta,\alpha} g_\beta$, all α, where the $r_{\beta,\alpha}$ are also polynomials in the $\{u_\delta\}$. Now these polynomials involve the $\{y_\gamma\}$ as parameters. Let $\{\bar{f}_\alpha\}$, $\{\bar{r}_{\beta,\alpha}\}$, $\{\bar{g}_\beta\}$ be the corresponding polynomials when the $\{y_\gamma\}$ are specialized to $\{w_\gamma\}$ say, where the $\{w_\gamma\}$ arise from a set $\{d_i\}$ of not necessarily invertible matrices. Then $\bar{f}_\alpha^q = \sum \bar{r}_{\beta,\alpha}\bar{g}_\beta$. But this means that all $\bar{f}_\alpha$ are zero if all $\bar{g}_\beta$ are zero. Referring to the meeting of $\{\bar{f}_\alpha\}$, $\{\bar{g}_\beta\}$, we conclude that whenever the first set of equations is satisfied by elements of a $p\times p$ matrix algebra, then so is the second, as promised. R is a subring of a direct sum of $p\times p$ matrix rings over fields, so the conclusion of the preceding sentence holds for R also. And in case the $\{d_i\}$ are now (ring) invertible elements of R, we can reverse the process and conclude that Q satisfies the standard identity of degree $2p$. Thus the theorem is proved.

To prove that Q satisfies the original multilinear homogeneous equation that R satisfied, consider R_1, which is R written with coefficients from the center F of Q. R_1 is prime, since R_1 has Q as a two-sided ring of quotients. R_1 is finite dimensional over F since Q is. Then R_1 is a finite dimensional simple algebra and hence is its own ring of quotients. So $R_1=Q$. But R_1 satisfies the original identity and therefore Q does.

Corollary. *Let A be an algebra satisfying a polynomial identity over its field and such that every element of A is a sum of nilpotent elements. Then A is nil.*

Proof. We show A has no prime quotients, thus proving A is its own lower nil radical. We remark that a prime ideal, in fact any ideal modulo which there are no nilpotent ideals, is an algebra ideal, so that a prime quotient R of A is also a polynomial identity algebra in which every element is a sum of nilpotents. But by the last part of the theorem, $(R_1=Q)$, the quotient matrix algebra Q of A also has this property, and the following known argument completes the proof. Q is a total matrix algebra over a division algebra D finite dimensional over its center F. Let K be a splitting field for D over F so that $Q_1=Q\otimes_F K$ is a total matrix algebra over K. Note that in Q_1, every element is still a sum of nilpotent elements, so that every element of Q_1 has trace zero. And yet the matrix with a 1 in the (1, 1) position and zeros elsewhere does not have trace zero. Thus the corollary is proven.

We remark that the corollary is true for rings with identities with one term having ± 1 as coefficient, but is false for arbitrary polynomial identity rings. For Harris [5] has produced a total 2×2 matrix ring over a division ring which we may take to be of characteristic 2 in which every element is a sum of nilpotent elements. The direct sum of this with a trivial algebra of characteristic 0 satisfies $2x_1x_2=0$ but is not nil.

Bibliography

1. S. Amitsur, *On rings with identities*, J. London Math. Soc. vol. 30 (1955) pp. 464–470.

2. A. W. Goldie, *The structure of prime rings under ascending chain conditions*, Proc. London Math. Soc. vol. 8 (1958) pp. 589–608.

3. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloquium Publications, vol. 37, 1956.

4. ———, *The theory of rings*, Mathematical Surveys, 1943.

5. B. Harris, *Commutators in divison rings*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 628–630.

University of Wisconsin, Madison