1. In [3], we have shown that in a finite $AB$-group $G$ in which $A$ and $B$ are cyclic and $A$ is its own normalizer, the commutator subgroup $T$ of $G$ is cyclic and $G = AT$ with $A \cap T = 1$. This result can be used to determine the structure of arbitrary $AB$-groups in which $A$ and $B$ are cyclic.

If $A$ is a subgroup of a group $G$, define the subgroup $N^i(A)$ of $G$ inductively by the formula $N^i(A) = N_i(N^{i-1}(A))$, and denote by $N^*(A)$ the upper bound of the subgroups $N^i(A)$. Using this notation, we shall prove the following theorem concerning $AB$-groups:

**Theorem A.** Let $G$ be a finite group of the form $AB$, where $A$ and $B$ are cyclic subgroups of $G$. Then $G$ contains a unique cyclic normal subgroup $T$ such that $G = N^*(A)T$ and $N^*(A) \cap T = 1$. Moreover, if $N^*(A) = AB^*$ with $B^* \subseteq B$, then $B^*$ and $T$ commute elementwise.

2. We begin with several lemmas:

**Lemma 1.** Let $G = AB$, with $A$ and $B$ cyclic, and assume that some subgroup $B'$ of $B$ is normal in $G$. Let $G = G/B' = \overline{A}\overline{B}$, where $\overline{A}$, $\overline{B}$ are the images of $A$, $B$ in $G$. Then $N^*(A)B'$ is the complete inverse image of $N^*(\overline{A})$ in $G$.

**Proof.** Let $B_0 \subseteq B'$ with $\sigma(B_0) = p$. Since $B'$ is cyclic, $B_0$ is normal in $G$. If $B_0 < B'$, set $G = G/B_0 = \overline{A}\overline{B}$, and let $\overline{B}'$ be the image of $B'$ in $G$. Since $G = G/B'$, it follows by induction on the order of $G$ that the inverse image of $N^*(\overline{A})$ in $G$ is $N^*(\overline{A})\overline{B}'$. Hence to prove the lemma, it suffices to show that $N^*(\overline{A})^{-1} = N^*(A)B_0$. Thus without loss of generality we may assume $\sigma(B') = p$.

Let $A = (a)$, $B = (b)$ and $B' = (b^*).$ It is clearly sufficient to prove by induction on $i$ that if $b^u \in N^i(\overline{A})^{-1}$, then $b^u \in N^i(A)B'$ for some $j$. Now for some integer $\lambda$ with $0 < \lambda < p$, we have

$$ab^*a^{-1} = b^{\lambda}.$$ 

We treat the cases $\lambda = 1$ and $\lambda > 1$ separately. If $\lambda = 1$, $B' \subseteq N(A)$. Now if $b^u \in N^i(\overline{A})^{-1}$, $b^u \in N^i(\overline{A})$ and hence $b^u(ab^*-u) \subseteq N^{i-1}(\overline{A})$. By induction $b^u(ab^*-u) \subseteq N^i(A)B'$. If $j = 0$, $b^u(ab^*-u) \subseteq N^i(A)$ and consequently $b^u \in N^2(A) = N^2_i(A)B'$. If $j > 0$, $N^i(A)B' = N^i(A)$, and so $b^u \in N^{i+1}(A) = N^{i+1}(A)B'$.
If \( \lambda > 1 \), it follows as above that \( b^u a b^{-u} \in N^i(A)B' \). If \( N^i(A) = AB_j \) and \( B_j = (b^j) \), we have

\[
(2) \quad b^u a b^{-u} = a^\alpha b^{\beta} b^{\gamma}
\]

for suitable integers \( \alpha, \beta, \gamma \).

Since \( (\lambda - 1, p) = 1 \), we can find an integer \( \delta \) such that \( \gamma + \delta \lambda \equiv \delta \pmod{p} \). We then have

\[
a^{-1}b^u + r \delta a = (a^{-1}b^u)(a^{-1}b^{r \delta}) = (a^{\alpha-1}b^{\beta+\gamma+u})b^{r \delta} = a^{\alpha-1}b^{\beta}b^{r \delta},
\]

whence \( b^u + r \delta \in N^i(A) \). Thus \( b^u \in N^i(A)B' \), and the lemma is proved.

**Lemma 2.** Let \( G = AB \), with \( A \) and \( B \) cyclic. Then \( G \) contains a cyclic subgroup \( T \), invariant under \( A \), such that \( G = N^*(A)T \) and \( N^*(A) \cap T = 1 \).

**Proof.** Either a subgroup of \( A \) or a subgroup of \( B \) is normal in \( G \) (Douglas [1]). Let \( A_1 \) be the maximal subgroup of \( A \) normal in \( G \), and assume first that \( A_1 \neq 1 \). If \( \overline{G} = G/A_1 = A\overline{B} \), we may assume by induction that \( \overline{G} = N^*(\overline{A})\overline{T} \), where \( N^*(\overline{A}) \cap \overline{T} = 1 \), \( \overline{T} \) is cyclic and invariant under \( \overline{A} \). Clearly \( N^*(A) = N^*(\overline{A})^{-1} \). If \( T_0 = T^{-1} \), \( G = N^*(A)T_0 \) where \( N^*(A) \cap T_0 = A_1 \) and \( T_0 \) is \( A \)-invariant.

If we let \( G_0 = AT_0 = AB_0 \) with \( B_0 \subset B \), it follows from our conditions that \( N_{G_0}(A) = A \). The proof of Theorem A of [3] now implies that if \( T = [G_0, G_0] \), then \( T \subset T_0 \), with \( T \) cyclic, \( G_0 = AT \) and \( A \cap T = 1 \). It follows at once that \( G = N^*(A)T \) with \( N^*(A) \cap T = 1 \), \( T \) cyclic and invariant under \( A \).

If \( A_1 = 1 \), we consider a minimal subgroup \( B' \) of \( B \) which is normal in \( G \); and this way we set \( \overline{G} = G/B' = A\overline{B} \). By induction \( \overline{G} = N^*(\overline{A})\overline{T} \), where \( \overline{T} \) is cyclic, \( \overline{A} \)-invariant, and \( N^*(\overline{A}) \cap \overline{T} = 1 \). If \( T_0 = \overline{T}^{-1} \), it follows from Lemma 1 that \( G = N^*(\overline{A})^{-1}T_0 = N^*(A)B'T_0 = N^*(A)T_0 \), where \( N^*(A) \cap T_0 = B' \). Using the notation of Lemma 1, we consider the cases \( \lambda = 1 \) and \( \lambda > 1 \) separately.

If \( \lambda = 1 \), set \( G_0 = AT_0 \) and \( \overline{G}_0 = A\overline{T} \). By Theorem A of [3], \( \overline{T} = [\overline{G}_0, \overline{G}_0] \). Hence we can find a commutator \( t \) in \( T_0 \), which maps on a generator \( \overline{t} \) of \( \overline{T} \). Let \( o(\overline{T}) = m \) and suppose, if possible, that \( t \) has order \( mp \). Since \( o(T_0) = mp \), it follows that \( T_0 = (t) \), and consequently \( T_0 = [G_0, G_0] \). If \( ata^{-1} = t^p \), \( [G_0, G_0] = (t^p^{-1}) \), and hence \( (\sigma - 1, mp) = 1 \). But \( t^m \in B' \) and, since \( \lambda = 1 \), \( B' \) is in the center of \( G \). Thus \( t^m = at^m a^{-1} = t \sigma \), whence \( p \mid (\sigma - 1) \), a contradiction.

If \( (t) = [G_0, G_0] \), we set \( T = (t) \). Since \( o(T) = m \), \( T \cap B' = 1 \). Furthermore \( T \) is normal in \( G_0 \). We conclude at once that \( G = N^*(A)T \), \( N^*(A) \cap T = 1 \), \( T \) cyclic and invariant under \( A \).

On the other hand, if \( (t) < [G_0, G_0] \), we must have \( T_0 = [G_0, G_0] \). Since \( G_0 \) is an \( AB \)-group, its commutator subgroup \( T_0 \) is abelian.
Now \( o(T_0) = mp \) and we have just shown that \( T_0 \) contains no commutator of order \( mp \). Therefore \( p \mid m \).

Since \( T_0 \) is normal in \( G_0 \) and is generated by \( t \) and \( b^r \), we have

\[
\tag{3} ata^{-1} = t^\sigma b^r \beta
\]

for suitable \( \sigma, \beta \).

It follows that \( \langle \bar{b}^{-1} \rangle = \langle \bar{G}_0, \bar{G}_0 \rangle \) and hence that \( (\sigma - 1, m) = 1 \). Since \( p \mid m \), there exists an integer \( \alpha \) such that \( \beta + \alpha = \sigma \alpha \) (mod \( p \)). Consequently \( a(tb^r) a^{-1} = t^\sigma b^\beta b^\alpha = t^\sigma b^{r+\alpha} = (tb^r)^\sigma \). It follows that the subgroup \( T = \langle tb^r \rangle \) is invariant under \( A \). Since \( o(T) = m \), \( T \cap B' = 1 \), and we conclude at once that \( G = N^*(A) \cup T, \) \( N^*(A) \cap T = 1 \), \( T \) cyclic and invariant under \( A \).

If \( \lambda > 1 \), we set \( P = \langle G_0, G_0 \rangle \). Since \( PCP \), \( A \ast (\mathcal{A}) \cap P' = CP \). Suppose, if possible, that \( B' \subset N^d(A) \), let \( d \) be the least integer such that \( B' \subset N^d(A) \). By definition of \( N^d(A) \), \( b^{r-1}b^{-r} \in N^{d-1}(A) \), and hence \( ab^{r-1}b^{-r} = b^{(\lambda - 1)} \in N^{d-1}(A) \). Since \( (\lambda - 1, p) = 1 \), it follows that \( b^r \in N^{d-1}(A) \), a contradiction. Thus \( B' \cap N^*(A) = 1 \), and consequently \( N^*(A) \cap T = 1 \). On the other hand, by Theorem A of [3], \( T \) is cyclic and \( G_0 = AB \). We conclude that in all cases \( G \) contains a cyclic subgroup \( T \), invariant under \( A \), such that \( G = N^*(A) \cup T \) and \( N^*(A) \cap T = 1 \).

**Lemma 3.** Let \( G = AB = N^*(A) \cup T \) with \( N^*(A) \cap T = 1 \), where \( T \) is cyclic and \( A \)-invariant and assume that \( A \cap B = 1 \). If \( N^*(A) = AB^* \) and \( AT = AB_0 \) with \( B^*, B_0 \subset B \), then \( (o(B^*), o(B_0)) = 1 \), \( B = B^* \times B_0 \), and \( o(T) = o(B_0) \).

**Proof.** \( G = N^*(A) \cup T = (AB^*) \cup T = (AB^*)(AT) = (AB^*)(AB_0) = A(B^*B_0) \). Since \( A \cap B = 1 \), it follows that \( B = B^*B_0 \). On the other hand, \( N^*(A) \cap T = 1 \), \( N^*(A) \cap AT = A \), and hence \( N^*(A) \cap B_0 \subset A \cap B_0 = 1 \). Thus \( B^* \cap B_0 = 1 \), whence \( B = B^* \times B_0 \). Since \( B^* \) and \( B_0 \) are subgroups of the cyclic group \( B \), it also follows that \( (o(B^*), o(B_0)) = 1 \).

Finally let \( T = \langle t \rangle \), where \( t = a^rb^r \) and let \( o(T) = m \). Since \( AT = AB_0 \), it follows as in the proof of Theorem 10 of [2] that \( T \) consists of the elements \( a^{qrb^r} \), and since \( A \cap B = 1 \), these elements must be distinct for \( j = 1, 2, \ldots, m \) and \( a^{nmrb^r} = 1 \). Hence \( b^{rm} = 1 \) and so \( o(B_0) \mid m \). On the other hand, if \( o(B_0) = n < m \), \( a^{nmrb^r} = a^{nm} \in A \cap T = 1 \), whence \( a^{nmrb^r} = a^{nmrb^r} \), a contradiction. Thus \( o(B_0) = o(T) \), as asserted.

**Lemma 4.** \( T \) is uniquely determined by the conditions \( G = AB = N^*(A) \cup T \) with \( N^*(A) \cap T = 1 \), \( T \) cyclic and \( A \)-invariant.

**Proof.** Suppose \( T, T' \) are two subgroups of \( G \) satisfying the conditions of the lemma. Let \( G_0 = AT \) and \( G_0' = AT' \). Since \( A \cap T = 1 \),
\( N_{\alpha}(A) = A \), whence by Theorem A of [3], \( T = [G_0, G_0] \), and similarly \( T' = [G'_0, G'_0] \). Hence to prove the lemma, it clearly suffices to show that \( G_0 = G'_0 \).

If \( A \cap B \neq 1 \), the equality of \( G_0 \) and \( G'_0 \) follows readily by induction by considering \( G = G/A \cap B \); hence without loss of generality we may assume that \( A \cap B = 1 \). If \( G_0 = AB_0 \) and \( G'_0 = AB'_0 \), it follows from Lemma 3 that \( B = B^* \times B_0 \) and \( B = B^* \times B'_0 \). Hence \( o(B_0) = o(B'_0) \). But \( B \), being cyclic, has a unique subgroup of any given order. Thus \( B_0 = B'_0 \) and \( G_0 = G'_0 \).

3. Proof of Theorem A. In view of Lemmas 2 and 4 it suffices to prove that \( T \) commutes elementwise with \( B^* \), for this will clearly imply that \( T \) is normal in \( G \). In this section we treat the case \( A \cap B = 1 \).

Let \( d \) be the least integer such that \( N_{d+1}(A) = N^d(A) \), so that \( N^*(A) = N^d(A) \). Let \( N_i(A) = AB_i \) with \( B_i = (b^r) \subset B \), \( i = 1, 2, \ldots, d \). Then \( B_1 < B_2 < \cdots < B_d \) and \( B_d = B^* \). We may assume \( r_i | r_{i-1} \), \( i = 2, 3, \ldots, d \). \( N_{i-1}(A) \) is normal in \( N_i(A) \) since \( N_{i-1}(A) = N_i(N_{i-1}(A)) \). Furthermore let \( G_0 = AT = AB_0 \) with \( B_0 = (b^r) \subset B \) and \( o(B_0) = m \). Then \( T = (t) \), where \( t = a^{s}b^r \) for some integer \( s \).

If \( s = 0 \), \( T = B_0 \), and it is obvious that \( T \) and \( B^* \) commute elementwise. Hence we may suppose \( s \neq 0 \) and without loss of generality that \( s | h \), where \( h = o(A) \). First of all, if \( h < sm \), \( a^{h}b^{h}s = b^{h}s \in T \), and generates a subgroup \( T_0 \), which is clearly invariant under \( B \) and hence is normal in \( G \). It follows at once by considering \( G/T_0 \) and using induction on the order of \( G \), that

\[
(t) b^{r}a^{t}b^{-r} = t b^{r(h/s)\beta}
\]

for some integer \( \beta \).

If \( n \) denotes the order of \( B^* \), we conclude at once from (4) that

\[
t = b^{r}a^{t}b^{-r} = b^{r(h/s)\beta n},
\]

whence

\[
(5) \quad r(h/s)\beta n \equiv 0 \pmod{m}.
\]

Since \( (n, m) = 1 \) by Lemma 3,

\[
r(h/s)\beta \equiv 0 \pmod{m} \quad \text{and} \quad b^{r}a^{t}b^{-r} = t, \text{as desired.}
\]

We may therefore assume that \( h = sm \). For \( i = 1, 2, \ldots, d \) we have

\[
(6) \quad b^{r}a^{t}b^{-r} = a^{u_{i-1}}b^{r_{i-1}+r_{i-1}^{-1}} \quad \text{for suitable integers} \ u_{i-1}, v_{i-1}, \text{where} \ r_{0} = 0.
\]

Let \( G'_i \) be the commutator subgroup of \( G_i = N_i(A)T \). We know that \( T = (t) \) is the commutator subgroup of \( G_0 = AT \). Since \( N_{i-1}(A) \) is normal in \( N_{i}(A) \), and \( G_i = N_i(A)B_0 \), \( G_{i-1} \) is normal in \( G_i \). It follows readily by induction that \( G'_i \) is generated by the elements \( a^{u_{i-1}}, \)
$a^{u_i-1}b^{r_i}v_i, \ldots, a^{u_i-1}b^{r_i}v_i-1, t$. Furthermore $G'_i$ is abelian since $G_i$ is an $AB$-group for each $i$.

To prove that $B^*$ and $T$ commute elementwise, we have only to show that $b^{r_i}t b^{-r_i} = t$ on the assumption that $b^{r_i}t b^{-r_i} = t$. Now from the form of $G'_i$, we have

(7) $b^{r_i}t b^{-r_i} = xt^r$ where $x \in N^{i-1}(A)$ and $xt = tx$.

Since by assumption $A \cap B = 1$, Lemma 3 implies $o(T) = o(B_0)$, whence $t^n = 1$. It follows now from (7) that $x^n = 1$. Suppose for some $j > 1$, $x \in N^j(A)$, $x \notin N^{j-1}(A)$. Let $\beta$ be the least integer such that $x^\beta \in N^{j-1}(A)$. Since $N^{j-1}(A)$ is normal in $N^j(A)$, $\beta \mid [N^j(A) : N^{j-1}(A)]$ and hence $\beta \mid o(B^*) = n$. But clearly $\beta \mid m$ since $x^n = 1$. Since $(n, m) = 1$, $\beta = 1$ and so $x \in N^{n-1}(A)$, a contradiction. Thus $x \in A$ and (7) takes the form

(8) $b^{r_i}t b^{-r_i} = a^\sigma t^\sigma, \quad a^\sigma t = t a^\sigma$.

Now $t = a^s b^r$ and $t^\gamma = a^s b^s \sigma$ for some integer $\sigma$, whence $b^{r_i} a^s b^{-r_i} = a^{s + \sigma r} b^{(r-1)s}$. But this implies $b^{(r-1)s} \in N^{n-1}(A) \cap B_0 = 1$, so that $\sigma \equiv 1 \pmod{m}$. Since $a^m = 1$, we may assume $\sigma = \gamma = 1$, and hence that

(9) $b^{r_i} a^s b^{-r_i} = a^{s+1}, \quad b^{r_i} t b^{-r_i} = a^s t$.

In particular, (9) implies that $s \mid \rho$.

Since $T$ is normal in $AT$, we have finally

(10) $a t a^{-1} = t^\lambda$ for some integer $\lambda$.

In view of (6)

(11) $(b^{r_i} a) t (b^{r_i} a)^{-1} = (a^{u_i-1} b^{r_i} v_i-1 t v_i-1)(a^{u_i-1} b^{r_i} v_i-1 t v_i-1)^{-1}$.

Using (9) and (10) and our assumption that $b^{r_i}$ commutes with $t$, we conclude readily from (11) that

(12) $a^s t^\lambda = a^s t^{s+1}$.

Since $A \cap T = 1$, $\rho (\lambda - 1) \equiv 0 \pmod{h}$. Since $h = ms$ and $s \mid \rho$, we obtain

(13) $\frac{\rho}{s} (\lambda - 1) \equiv 0 \pmod{m}$.

But $T$ is the commutator subgroup of $AT$, which implies $(\lambda - 1, m) = 1$; and it follows from (13) that $\rho \equiv 0 \pmod{h}$. Hence $b^{r_i} t b^{-r_i} = t$, as desired. We conclude that $B^*$ and $T$ commute elementwise.

4. Finally we treat the case $A \cap B \neq 1$. Let $\bar{G} = G/A \cap B = \bar{A} \bar{B}$
\[ N^*(\overline{A}) \overline{T}, \text{ where } \overline{A}, \overline{B}, \overline{T} \text{ are the images of } A, B, T, \text{ in } \overline{G}. \] Clearly \( \overline{T} \) is cyclic, invariant under \( \overline{A} \), and \( N^*(\overline{A}) \cap \overline{T} = 1 \). If \( N^*(\overline{A}) = \overline{AB}^* \) with \( \overline{B}^* \subseteq \overline{B} \), let \( n = o(\overline{B}^*) \); and let \( m = o(\overline{T}) \). Since \( A \cap B = 1 \), it follows from the preceding section that \( (n, m) = 1 \) and that \( \overline{B}^* \) commutes elementwise with \( \overline{T} \). Furthermore \( N^*(\overline{A})^{-1} = N^*(A)(A \cap B) = N^*(A) \), and hence \( B^* \) is the inverse image of \( \overline{B}^* \) in \( G \). If \( B^* = (b^d) \), it follows that

\[ b^xtb^{-td} = xt, \quad x \in A \cap B, \quad \text{and} \quad b^{ran} \in A \cap B. \]

Since \( A \cap B \) is in the center of \( G \), \( x^n = 1 \). On the other hand \( (14) \) yields \( t = b^{ran}tb^{-ran} = x^n t \), whence \( x^n = 1 \). Since \( (n, m) = 1 \), we conclude that \( x = 1 \); and the theorem is proved.

Bibliography


Clark University and
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