UNIVALENCE OF BESSEL FUNCTIONS

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1. **Introduction.** In 1954 M. S. Robertson [2] obtained sufficient conditions for the univalence in the unit circle of functions

$$[W(z)]^{1/\alpha} = \left[z^{\alpha} \sum_{n=0}^{\infty} a_n z^n \right]^{1/\alpha}, \qquad a_0 = 1,$$

where $\Re\{\alpha\} \geqq 1/2$ and $W(z)$ is a solution of the differential equation

$$W''(z) + p(z)W(z) = 0, \qquad |z| < 1. \tag{1.1}$$

In this paper we employ the methods of Robertson to obtain information concerning the univalence of the functions $[T(z)]^{1/\nu}$ $(\nu \neq 0)$ and $z^{1-\nu}T(z)$ where

$$T(z) = z^{\nu} \sum_{n=0}^{\infty} a_n z^n, \qquad \Re\{\nu\} \geqq 0,$$

is a solution of the differential equation

$$T''(z) + \frac{1}{z} T'(z) + q(z)T(z) = 0, \qquad |z| < R. \tag{1.2}$$

In particular we shall first determine a radius of univalence for the normalized Bessel functions $[J_\nu(z)]^{1/\nu}$ for values of ν belonging to the region G defined by the inequalities $\Re\{\nu\} > 0$, $|\arg \nu| < \pi/4$. Then we shall determine the radius of univalence of the functions $z^{1-\nu}J_\nu(z)$ for values of ν belonging to a subset of the closure of G. When ν is real and positive we shall determine the exact radius of star-likeness of both of the above-mentioned classes of normalized Bessel functions.

Our results concerning the functions $z^{1-\nu}J_\nu(z)$ "sharpen" those of Kreyszig and Todd [1] when $\nu \geqq 0$ and extend their results for complex values of ν.

2. **Preliminaries.** Let

$$z^2 p^*(z) = \sum_{n=0}^{\infty} \overset{*}{p}_n z^n, \qquad \overset{*}{p}_0 \leqq \frac{1}{4}, \tag{2.1}$$

be regular for $|z| < R$ and real on the real axis. Given any nonnegative constant C, define

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$$W_C(z) = z^{\alpha^*} \sum_{n=0}^{\infty} \overset{*}{a}_n(C) z^n, \qquad \overset{*}{a}_0(C) = 1, \tag{2.2}$$

to be the unique solution for $|z| < R$ of the differential equation

$$W''(z) + \{C[p^*(z) - \overset{*}{p}_0/z^2] + \overset{*}{p}_0/z^2\} W(z) = 0 \tag{2.3}$$

where α^* is the larger root of the associated indicial equation.

We now are ready to state the following

LEMMA. *Let $y(\rho)$, $dy(\rho)/d\rho = y'(\rho)$ be real functions, continuous in the real variable ρ for $0 < \rho < R$. For small values of ρ let*

$$y(\rho) = O(\rho^\delta), \qquad y'(\rho) = O(\rho^{\delta-1}), \qquad \delta > 1/2.$$

Then

$$\begin{aligned} &\int_0^r \{C[\rho^2 p^*(\rho) - \overset{*}{p}_0] + \overset{*}{p}_0\} y^2(\rho) \frac{d\rho}{\rho^2} \\ &\qquad \leqq \int_0^r [y'(\rho)]^2 d\rho - \frac{W_C'(r)}{W_C(r)} \cdot y^2(r), \qquad 0 < r < R, \end{aligned} \tag{2.4}$$

where $W_C(z)$ is the solution (2.2) *of* (2.3).

The proof of this lemma is with obvious modifications the same as that given by Robertson [2] for the case $R = 1$. We will not reproduce it here.

With the aid of the lemma we are able to prove the following

THEOREM 1. *Let $z^2 p(z)$ be regular for $|z| < R$ and satisfy the inequality*

$$\mathcal{R}\{e^{i\gamma} z^2 p(z)\} \leqq \cos\gamma \{C[|z|^2 p^*(|z|) - \overset{*}{p}_0] + \overset{*}{p}_0\} \tag{2.5}$$

where $C \geqq 0$, $|\gamma| \leqq \pi/2$, and $z^2 p^*(z)$ is defined in (2.1). *With $p(z)$ chosen in this manner we define*

$$W(z) = z^\alpha \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 = 1, \quad |z| < R, \tag{2.6}$$

to be the unique solution of (1.1) *corresponding to the root with larger real part of the associated indicial equation. Let $W_C(z)$ be defined as in* (2.2). *Then*

$$\mathcal{R}\left\{e^{i\gamma} \frac{zW'(z)}{W(z)}\right\} \geqq |z| \frac{W_C'(|z|)}{W_C(|z|)} \cos\gamma \tag{2.7}$$

for all $|z| \leqq \rho < R$.

PROOF. If in (2.6) we have $\mathcal{R}\{\alpha\} > 1/2$ then

$$
\begin{aligned}
&| W(z) |^2 \Re\left\{ e^{i\gamma} \frac{zW'(z)}{W(z)} \right\} \\
(2.8)\qquad &= r\cos\gamma \int_0^r | W' |^2 d\rho - r\cdot \int_0^r \Re\{ e^{i\gamma} z^2 p(z)\}_{|z|=\rho} \frac{| W |^2}{\rho^2} d\rho
\end{aligned}
$$

for all $0 \leqq r < R$.

This equation is known as the "Green's transform" of (1.1) and in the form (2.8) is due to Robertson [**2**]. The inequality (2.7) now follows immediately from (2.8), (2.5), and (2.4) with $y(\rho) = W(\rho)$.

The proof for the case when $\Re\{\alpha\} = 1/2$ in (2.6) follows from the continuity of $zW'(z)/W(z)$ as a function of α for $\Re\{\alpha\} > 0$ (see [**2**, p. 258]).

We conclude this section with the definitions of the terms "star-like" and "spiral-like."

Definition. A function $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 \neq 0$, regular for $|z| < R$ will be called *spiral-like* in $|z| < R$ if for some real constant $\gamma(|\gamma| \leqq \pi/2)$ the function $f(z)$ satisfies the inequality

$$
(2.9)\qquad \Re\left\{ e^{i\gamma} \frac{zf'(z)}{f(z)} \right\} \geqq 0
$$

for all $|z| < R$. In the special case when $\gamma = 0$ we say that $f(z)$ is *star-like* with respect to the origin in $|z| < R$.

It was shown by Špaček [**3**] that (2.9) is sufficient for the univalence in $|z| < R$ of $f(z)$ whenever $f'(0) \neq 0$.

3. **Bessel's equation.** In this section we state our two theorems concerning the univalence of normalized solutions of Bessel's equation

$$
(3.1)\qquad T''(z) + \frac{1}{z} T'(z) + \left(1 - \frac{\nu^2}{z^2}\right) T(z) = 0, \qquad | z | < R.
$$

Theorem 2. *Let the complex number ν satisfy the inequalities $\Re\{\nu\} > 0$, $|\arg \nu| < \pi/4$. Then the normalized Bessel function $[J_\nu(z)]^{1/\nu}$ is regular, univalent, and spiral-like in every circle $|z| = r < \rho_\mu$ where $\mu^2 = \Re\{\nu^2\}$, $\mu > 0$, and ρ_μ is the smallest positive zero of the function $J'_\mu(r)$. In the particular case when ν is real and positive the function $[J_\nu(z)]^{1/\nu}$ is star-like in $|z| < \rho_\mu$ but is not univalent in any larger circle.*

Theorem 3. *Let the complex number $\nu = x + iy$ satisfy one of the following conditions*:

$$
(3.2)\qquad 0 \leqq x < 1 \quad \textit{and} \quad y \leqq x,
$$

$$
(3.3)\qquad x \geqq 1 \quad \textit{and} \quad y^2 < 2x - 1.
$$

Then the normalized Bessel function $z^{1-\nu}J_\nu(z)$ is regular, univalent, and spiral-like in every circle $|z| = r < \rho_\mu^*$ where $\mu^2 = \Re\{\nu^2\}$, $\mu > 0$, and ρ_μ^* is the smallest positive zero of the function $rJ_\mu'(r) + \Re\{1-\nu\}J_\mu(r)$. In the particular case when ν is real the function $z^{1-\nu}J_\nu(z)$ is univalently star-like in $|z| < \rho_\mu^*$ but is not univalent in any larger circle.

4. **Proof of Theorem** 2. Select any ν satisfying the inequalities $\Re\{\nu\} > 0$, $|\arg \nu| < \pi/4$, and consider (2.3) with $\mu^2 = \Re\{\nu^2\}$, $C=1$, and

$$z^2p^*(z) = z^2 + 1/4 - \mu^2. \tag{4.1}$$

In this manner we obtain the differential equation

$$W''(z) + \left[1 - \frac{1}{z^2}(\mu^2 - 1/4)\right]W(z) = 0 \tag{4.2}$$

whose solution $W_1(z)$ as defined in (2.2) is

$$W_1(z) = 2^\mu\Gamma(\mu + 1)z^{1/2}J_\mu(z). \tag{4.3}$$

Next, by setting

$$z^2p(z) = z^2 + 1/4 - \mu^2 \tag{4.4}$$

in (1.1) we find that the solution $W(z)$ of (1.1) as defined in (2.1) is

$$W(z) = 2^\nu\Gamma(\nu + 1)z^{1/2}J_\nu(z). \tag{4.5}$$

The solutions given in (4.3) and (4.5) are valid for all finite z. Moreover, $z^2p^*(z)$ and $z^2p(z)$ as given in (4.1) and (4.4) satisfy (2.7) for all finite values of z when $\gamma = 0$ and $C=1$. Therefore, from (2.7) it follows that

$$\Re\left\{\frac{zW'(z)}{W(z)}\right\} \geqq |z| \frac{W_1'(|z|)}{W_1(|z|)} \tag{4.6}$$

for all finite values of z. Thus, from (4.3), (4.5), and (4.6) we have

$$\Re\left\{\frac{zJ_\nu'(z)}{J_\nu(z)}\right\} \geqq \frac{rJ_\mu'(r)}{J_\mu(r)}, \qquad |z| = r, \tag{4.7}$$

for all finite r.

Since μ is positive it follows from (4.7) that

$$\Re\left\{\frac{zJ_\mu'(z)}{J_\mu(z)}\right\} \geqq 0, \qquad |z| \leqq \rho_\mu, \tag{4.8}$$

where ρ_μ is the smallest positive zero of $J_\mu'(r)$.

We now define

$$F_\nu(z) = [J_\nu(z)]^{1/\nu} \tag{4.9}$$

where $[J_\nu(z)]^{1/\nu} = (1/\nu)\exp\{\operatorname{Log} J_\nu(z)\}$ and Log represents the principal branch of the logarithm. Then,

$$\mathcal{R}\left\{\frac{zF_\nu'(z)}{F_\nu(z)}\right\} = \mathcal{R}\left\{\frac{zJ_\nu'(z)}{J_\nu(z)}\right\} \tag{4.10}$$

and it follows from (4.8) that in every circle $|z| = r < \rho_\mu$ the function $F_\nu(z)$ is spiral-like if ν is complex and is star-like if ν is real and positive.

Clearly, since $J_\mu'(z)$ vanishes for $z = \rho_\mu$ the function $[J_\mu(z)]^{1/\mu}$, $\mu > 0$, cannot be univalent in any circle $|z| = r > \rho_\mu$.

This completes the proof of Theorem 2.

5. **Proof of Theorem** 3. If in the proof of Theorem 2 we replace $F_\nu(z)$ in (4.9) by the function

$$S_\nu(z) = z^{1-\nu} J_\nu(z), \qquad \mathcal{R}\{\nu\} \geqq 0, \tag{5.1}$$

then since

$$\mathcal{R}\left\{\frac{zS_\nu'(z)}{S_\nu(z)}\right\} = \mathcal{R}\{1-\nu\} + \mathcal{R}\left\{\frac{zJ_\nu'(z)}{J_\nu(z)}\right\} \tag{5.2}$$

it follows from (4.7) that

$$\mathcal{R}\left\{\frac{zS_\nu'(z)}{S_\nu(z)}\right\} \geqq \mathcal{R}\{1-\nu\} + \frac{zJ_\mu'(z)}{J_\mu(z)} \tag{5.3}$$

for all finite z ($|z| = r$). Then, since (3.2) and (3.3) imply that the right-hand member of (5.3) is positive for sufficiently small values of r, it follows that

$$\mathcal{R}\left\{\frac{zS_\nu'(z)}{S_\nu(z)}\right\} \geqq 0, \qquad |z| \leqq \rho_\mu^*, \tag{5.4}$$

where ρ_μ^* is the smallest positive zero of the function

$$rJ_\mu'(r) + \mathcal{R}\{1-\nu\} J_\mu(r).$$

For non-negative real values of ν the vanishing of $S_\nu'(z)$ for $z = \rho_\mu^*$ precludes the possibility that $S_\nu(z)$ is univalent in any circle $|z| = r > \rho_\mu^*$.

We note here that for non-negative real values of ν the ρ_μ^* of our Theorem 3 is precisely the ρ_ν of [1].

6. **Remarks.** We note that if $T(z)=z^\nu \sum_{n=0}^\infty a_n z^n$, $\Re\{\nu\} \geqq 0$, satisfies (1.2) for $|z| < R$, then the function

$$W(z) = z^{1/2}T(z) = z^\alpha \sum_{n=0}^{\infty} a_n z^n, \qquad \Re\{\alpha\} \geqq 1/2,$$

satisfies (1.1) with $z^2p(z) = z^2q(z) + 1/4$. Thus Theorem 1 is applicable to an entire class of functions satisfying (1.2). In particular, therefore, one could obtain results analogous to those of Theorems 2 and 3 for the modified Bessel functions $I_\nu(z)$.

Many other results could be obtained by judicious selection of the function $q(z)$ subject to the conditions of Theorem 1.

Bibliography

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