UNIVALENCE OF BESSEL FUNCTIONS

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1. Introduction. In 1954 M. S. Robertson [2] obtained sufficient conditions for the univalence in the unit circle of functions

\[ [W(z)]^{1/\alpha} = \left[ z^\alpha \sum_{n=0}^{\infty} a_n z^n \right]^{1/\alpha}, \quad a_0 = 1, \]

where \( \Re \{ \alpha \} \geq 1/2 \) and \( W(z) \) is a solution of the differential equation

\[ W''(z) + p(z)W(z) = 0, \quad |z| < 1. \] \( (1.1) \)

In this paper we employ the methods of Robertson to obtain information concerning the univalence of the functions \([T(z)]^{1/\nu} (\nu \neq 0)\) and \( z^{1-\nu}T(z) \) where

\[ T(z) = z^\nu \sum_{n=0}^{\infty} a_n z^n, \quad \Re \{ \nu \} \geq 0, \]

is a solution of the differential equation

\[ T''(z) + \frac{1}{z} T'(z) + q(z)T(z) = 0, \quad |z| < R. \] \( (1.2) \)

In particular we shall first determine a radius of univalence for the normalized Bessel functions \([J_\nu(z)]^{1/\nu}\) for values of \( \nu \) belonging to the region \( G \) defined by the inequalities \( \Re \{ \nu \} > 0, |\arg \nu| < \pi/4 \). Then we shall determine the radius of univalence of the functions \( z^{1-\nu}J_\nu(z) \) for values of \( \nu \) belonging to a subset of the closure of \( G \). When \( \nu \) is real and positive we shall determine the exact radius of star-likeness of both of the above-mentioned classes of normalized Bessel functions.

Our results concerning the functions \( z^{1-\nu}J_\nu(z) \) "sharpen" those of Kreyszig and Todd [1] when \( \nu \geq 0 \) and extend their results for complex values of \( \nu \).

2. Preliminaries. Let

\[ z^2 p^*(z) = \sum_{n=0}^{\infty} p_n z^n, \quad p_0^* \leq \frac{1}{4}, \] \( (2.1) \)

be regular for \( |z| < R \) and real on the real axis. Given any non-negative constant \( C \), define
(2.2) \[ W_C(z) = z^{\alpha^*} \sum_{n=0}^{\infty} a_n^*(C)z^n, \quad a_0^*(C) = 1, \]

to be the unique solution for \(|z| < R\) of the differential equation

(2.3) \[ W''(z) + \left\{ C[p^*(z) - p_0^*/z^2] + p_p^*/z^2 \right\} W(z) = 0 \]

where \(\alpha^*\) is the larger root of the associated indicial equation.

We now are ready to state the following

**Lemma.** Let \(y(\rho), dy(\rho)/d\rho = y'(\rho)\) be real functions, continuous in the real variable \(\rho\) for \(0 < \rho < R\). For small values of \(\rho\) let \(y(\rho) - O(\rho^\delta), \quad y'(\rho) = O(\rho^{\delta-1}), \quad \delta > 1/2.\)

Then

\[
\int_0^r \left\{ C[p^2p^*(\rho) - p_0^*] + p_p^* \right\} y^2(\rho) \frac{d\rho}{\rho^2} \leq \int_0^r \left[ y'(\rho) \right]^2 d\rho - \frac{W_C'(r)}{W_C(r)} \frac{W_c(\rho)}{W_c(\rho)} \quad 0 < r < R,
\]

where \(W_C(z)\) is the solution (2.2) of (2.3).

The proof of this lemma is with obvious modifications the same as that given by Robertson [2] for the case \(R = 1\). We will not reproduce it here.

With the aid of the lemma we are able to prove the following

**Theorem 1.** Let \(z^2p(z)\) be regular for \(|z| < R\) and satisfy the inequality

(2.5) \[ \Re\{e^{i\gamma}z^2p(z)\} \leq \cos \gamma \{ C[|z|^2p^*(|z|) - p_0^*] + p_p^* \} \]

where \(C \geq 0, \quad |\gamma| \leq \pi/2,\) and \(z^2p^*(z)\) is defined in (2.1). With \(p(z)\) chosen in this manner we define

(2.6) \[ W(z) = z^\alpha \sum_{n=0}^{\infty} a_nz^n, \quad a_0 = 1, \quad |z| < R, \]

to be the unique solution of (1.1) corresponding to the root with larger real part of the associated indicial equation. Let \(W_C(z)\) be defined as in (2.2). Then

(2.7) \[ \Re \left\{ e^{i\gamma} \frac{zW'(z)}{W(z)} \right\} \geq |z| \frac{W_C'(|z|)}{W_C(|z|)} \cos \gamma \]

for all \(|z| \leq \rho < R.\)

**Proof.** If in (2.6) we have \(\Re\{\alpha\} > 1/2\) then

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This equation is known as the “Green’s transform” of (1.1) and in the form (2.8) is due to Robertson [2]. The inequality (2.7) now follows immediately from (2.8), (2.5), and (2.4) with \( \gamma(\rho) = W(\rho) \).

The proof for the case when \( \Re \{\alpha\} = 1/2 \) in (2.6) follows from the continuity of \( zW'(z)/W(z) \) as a function of \( \alpha \) for \( \Re \{\alpha\} > 0 \) (see [2, p. 258]).

We conclude this section with the definitions of the terms “star-like” and “spiral-like.”

**Definition.** A function \( f(z) = \sum_{n=1}^{\infty} a_n z^n, a_1 \neq 0 \), regular for \( |z| < R \), will be called spiral-like in \( |z| < R \) if for some real constant \( \gamma(\gamma) = \pi/2 \) the function \( f(z) \) satisfies the inequality

\[
\Re \left\{ e^{i\gamma} \frac{zf'(z)}{f(z)} \right\} \geq 0
\]

for all \( |z| < R \). In the special case when \( \gamma = 0 \) we say that \( f(z) \) is star-like with respect to the origin in \( |z| < R \).

It was shown by Špaček [3] that (2.9) is sufficient for the univalence in \( |z| < R \) of \( f(z) \) whenever \( f'(0) \neq 0 \).

3. **Bessel’s equation.** In this section we state our two theorems concerning the univalence of normalized solutions of Bessel’s equation

\[
T''(z) + \frac{1}{z} T'(z) + \left( 1 - \frac{\nu^2}{z^2} \right) T(z) = 0, \quad |z| < R.
\]

**Theorem 2.** Let the complex number \( \nu \) satisfy the inequalities \( \Re \{\nu\} > 0, |\arg \nu| < \pi/4 \). Then the normalized Bessel function \( [J_\nu(z)]^{1/\nu} \) is regular, univalent, and spiral-like in every circle \( |z| = r < \rho_\mu \) where \( \mu^2 = \Re \{\nu^2\}, \mu > 0 \), and \( \rho_\mu \) is the smallest positive zero of the function \( J_\nu(r) \). In the particular case when \( \nu \) is real and positive the function \( [J_\nu(z)]^{1/\nu} \) is star-like in \( |z| < \rho_\mu \) but is not univalent in any larger circle.

**Theorem 3.** Let the complex number \( \nu = x + iy \) satisfy one of the following conditions:

\[
\begin{align*}
(3.2) \quad & 0 \leq x < 1 \quad \text{and} \quad y \leq x, \\
(3.3) \quad & x \geq 1 \quad \text{and} \quad y^2 < 2x - 1.
\end{align*}
\]
Then the normalized Bessel function $z^{1-v}J_\nu(z)$ is regular, univalent, and spiral-like in every circle $|z| = r < \rho^*_\mu$ where $\mu^2 = \Re \{\nu^2\}$, $\mu > 0$, and $\rho^*_\mu$ is the smallest positive zero of the function $rJ'_\mu(r) + \Re \{1-v\} J_\mu(r)$. In the particular case when $\nu$ is real the function $z^{1-v}J_\nu(z)$ is univalently star-like in $|z| < \rho^*_\mu$ but is not univalent in any larger circle.

4. Proof of Theorem 2. Select any $\nu$ satisfying the inequalities $\Re \{\nu\} > 0$, $|\arg \nu| < \pi/4$, and consider (2.3) with $\mu^2 = \Re \{\nu^2\}$, $C = 1$, and

\[(4.1)\quad z^2 \rho^*(z) = z^2 + 1/4 - \mu^2.\]

In this manner we obtain the differential equation

\[(4.2)\quad W''(z) + \left[1 - \frac{1}{z^2} (\mu^2 - 1/4)\right] W(z) = 0\]

whose solution $W_1(z)$ as defined in (2.2) is

\[(4.3)\quad W_1(z) = 2^{\nu} \Gamma(\nu + 1) z^{1/2} J_\nu(z).\]

Next, by setting

\[(4.4)\quad z^2 \rho(z) = z^2 + 1/4 - \mu^2\]

in (1.1) we find that the solution $W(z)$ of (1.1) as defined in (2.1) is

\[(4.5)\quad W(z) = 2^{\nu} \Gamma(\nu + 1) z^{1/2} J_\nu(z).\]

The solutions given in (4.3) and (4.5) are valid for all finite $z$. Moreover, $z^2 \rho^*(z)$ and $z^2 \rho(z)$ as given in (4.1) and (4.4) satisfy (2.7) for all finite values of $z$ when $\gamma = 0$ and $C = 1$. Therefore, from (2.7) it follows that

\[(4.6)\quad \Re \left\{ \frac{zW'(z)}{W(z)} \right\} \geq \left| z \right| \frac{W'_1 \left( \left| z \right| \right)}{W_1 \left( \left| z \right| \right)}\]

for all finite values of $z$. Thus, from (4.3), (4.5), and (4.6) we have

\[(4.7)\quad \Re \left\{ \frac{zJ'_\nu(z)}{J_\nu(z)} \right\} \geq \frac{rJ'_\mu(r)}{J_\mu(r)}, \quad |z| = r,\]

for all finite $r$.

Since $\mu$ is positive it follows from (4.7) that

\[(4.8)\quad \Re \left\{ \frac{zJ'_\nu(z)}{J_\nu(z)} \right\} \geq 0, \quad |z| \leq \rho_\mu,\]

where $\rho_\mu$ is the smallest positive zero of $J'_\mu(r)$.
We now define

\[
F_\nu(z) = [J_\nu(z)]^{1/\nu}
\]

where \([J_\nu(z)]^{1/\nu} = (1/\nu) \exp \{ \log J_\nu(z) \}\) and \(\log\) represents the principal branch of the logarithm. Then,

\[
\Re \left\{ \frac{zF_\nu'(z)}{F_\nu(z)} \right\} = \Re \left\{ \frac{zJ_\nu'(z)}{J_\nu(z)} \right\}
\]

and it follows from (4.8) that in every circle \(|z| = r < \rho_\mu\) the function \(F_\nu(z)\) is spiral-like if \(\nu\) is complex and is star-like if \(\nu\) is real and positive.

Clearly, since \(J_\nu'(z)\) vanishes for \(z = \rho_\mu\) the function \([J_\nu(z)]^{1/\nu}, \mu > 0,\) cannot be univalent in any circle \(|z| = r > \rho_\mu\).

This completes the proof of Theorem 2.

5. Proof of Theorem 3. If in the proof of Theorem 2 we replace \(F_\nu(z)\) in (4.9) by the function

\[
S_\nu(z) = z^{1-\nu}J_\nu(z), \quad \Re \{\nu\} \geq 0,
\]

then since

\[
\Re \left\{ \frac{zS_\nu'(z)}{S_\nu(z)} \right\} = \Re \{1 - \nu\} + \Re \left\{ \frac{zJ_\nu'(z)}{J_\nu(z)} \right\}
\]

it follows from (4.7) that

\[
\Re \left\{ \frac{zS_\nu'(z)}{S_\nu(z)} \right\} \geq \Re \{1 - \nu\} + \frac{zJ_\nu'(z)}{J_\nu(z)}
\]

for all finite \(z (|z| = r)\). Then, since (3.2) and (3.3) imply that the right-hand member of (5.3) is positive for sufficiently small values of \(r\), it follows that

\[
\Re \left\{ \frac{zS_\nu'(z)}{S_\nu(z)} \right\} \geq 0, \quad |z| \leq \rho_\mu^*,
\]

where \(\rho_\mu^*\) is the smallest positive zero of the function

\[
rJ_\nu'(r) + \Re \{1 - \nu\} J_\nu(r).
\]

For non-negative real values of \(\nu\) the vanishing of \(S_\nu(z)\) for \(z = \rho_\mu^*\) precludes the possibility that \(S_\nu(z)\) is univalent in any circle \(|z| = r > \rho_\mu^*\).

We note here that for non-negative real values of \(\nu\) the \(\rho_\mu^*\) of our Theorem 3 is precisely the \(\rho_\nu\) of [1].
6. Remarks. We note that if \( T(z) = z^\nu \sum_{n=0}^{\infty} a_n z^n \), \( \Re\{\nu\} \geq 0 \), satisfies (1.2) for \( |z| < R \), then the function
\[
W(z) = z^{1/2} T(z) = z^\alpha \sum_{n=0}^{\infty} a_n z^n, \quad \Re\{\alpha\} \geq 1/2,
\]
satisfies (1.1) with \( z^2 p(z) = z^2 q(z) + 1/4 \). Thus Theorem 1 is applicable to an entire class of functions satisfying (1.2). In particular, therefore, one could obtain results analogous to those of Theorems 2 and 3 for the modified Bessel functions \( I_\nu(z) \).

Many other results could be obtained by judicious selection of the function \( q(z) \) subject to the conditions of Theorem 1.

**Bibliography**


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