of \( z \) and each \( S_j \) is even. Let \( s \) be the number of points of the orbit and \( q \) any index in \( Q \). For \( r < s \), \((hz)^r(\phi, q)\) differs from \((\phi, q)\) in the first coordinate; but \((hz)^s(\phi, q) = (\phi, q)\). Thus every element of \( G \) has an odd cycle. As we noted above, this implies \([1]\) the existence of a fair game of \( 2^k(2^l - 1) \) players.

**References**


**University of Washington**

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**ON INDUCED TOPOLOGIES IN QUASI-REFLEXIVE BANACH SPACES**

**LARRY C. HUNTER**

1. **Introduction.** Let \( \pi \) denote the canonical isomorphism of a Banach space \( X \) into its second conjugate space \( X^{**} \). An example is given by James \([4]\) of a space \( X \) for which \( X \) is separable, \( X \) is not reflexive, \( X \) is isomorphic to \( X^{**} \), and \( X^{**}/\pi(X) \) is one-dimensional. Civin and Yood undertook a more complete investigation of Banach spaces \( X \) such that \( X^{**}/\pi(X) \) is (finite) \( n \)-dimensional and called such spaces quasi-reflexive Banach spaces of order \( n \). If \( Q \) is a subset of \( X^* \), let \( \sigma(X, Q) \) denote the least fine topology for \( X \) such that all \( x^* \in Q \) are continuous. In \([1]\) Civin and Yood establish the following result.

**Theorem A.** The following statements are equivalent:

1. \( X \) is quasi-reflexive of order \( n \).
2. There is an equivalent norm for \( X \) such that \( X^* = Q \oplus R \) where \( Q \) is a total closed linear manifold such that the unit ball of \( X \) is compact in \( \sigma(X, Q) \) and \( R \) is an \( n \)-dimensional linear manifold.

It is the purpose of this paper to study properties of the topologies \( \sigma(X, Q) \), where \( X^* = Q \oplus R \), \( Q \) is a total closed linear manifold, and

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$R$ is $n$-dimensional. It is shown that $\sigma(X, Q)$ is nothing more than the $w^*$-topology on $X$ when $X$ is considered as the conjugate space of $Q$.

2. Notation. Let $X$ be a Banach space. Let $\pi$ be the canonical isomorphism of $X$ into $X^{**}$, its second conjugate space. For a subset $A$ of $X$, $A^+$ will designate the annihilator of $A$ in $X^*$, and $A^{++}$ the annihilator of $A^+$ in $X^{**}$. For a set $B$ in $X^*$, $B^-$ will denote the annihilator of $B$ in $X$. When we write $X = C \oplus D$, we shall mean that $C$ and $D$ are closed linear manifolds of $X$, that $X$ is the linear span of $C$ and $D$, and $C \cap D = 0$. We define $S_r = \{ x \in X : \|x\| \leq r \}$.

3. Preliminary results. If $X$ is a quasi-reflexive Banach space of order $n$, then $X^{**} = \pi(X) \oplus L$ where $L$ is an $n$-dimensional linear manifold. Civin and Yood note that $X^* = Q \oplus R$ where $Q = L^*$ is total and $R$ is $n$-dimensional. In the proof of Theorem A, they show that for $Q = L^*$ there is an equivalent norm for $X$ such that the unit ball of $X$ is compact in $\sigma(X, Q)$.

The following question can then be posed. If $X$ is a quasi-reflexive space of order $n$ and $X^* = Q_0 \oplus R_0$ where $Q_0$ is total and $R_0$ is $n$-dimensional, is there an equivalent norm for $X$ in which the unit ball is compact in $\sigma(X, Q_0)$? The following theorem shows that all decompositions of $X^*$ of the above type arise from considering the annihilators of the $n$-dimensional pieces of the second conjugate space of $X$.

3.1. Theorem. If $X$ is a quasi-reflexive Banach space of order $n$ and if $X^* = Q_0 \oplus S_0$ where $Q_0$ is total and $S_0$ is $n$-dimensional, then:

(i) $X^{**} = \pi(X) \oplus Q_0^+$,
(ii) there is an equivalent norm for $X$ such that the unit ball of $X$ is compact in $\sigma(X, Q_0)$,
(iii) $\|x\| = \sup_{x^* \in Q_0, \|x^*\| = 1} |x^*(x)|$ if $X$ has the norm for which the unit ball is compact in $\sigma(X, Q)$.

Proof. (i) Suppose that $x^{**} \in \pi(X) \cap Q_0^+$. Then $x^{**} = \pi(x)$ for some $x \in X$ and for all $y^* \in Q_0$, $x^{**}(y^*) = y^*(x) = 0$. Since $Q_0$ is total, $x = 0$, and hence $\pi(X) \cap Q_0^+ = 0$. Since $X^{**}/\pi(X)$ has dimension $n$, it follows that $Q_0^+$ has dimension $r \leq n$. Let $x_1^{**}, x_2^{**}, \ldots, x_r^{**}$ be a basis for $Q_0^+$ and select $x_1^*, x_2^*, \ldots, x_r^* \in X^*$ such that $x_i^{**}(x_j^*) = \delta_{ij}$, $i, j = 1, 2, \ldots, r$. Let $R$ be the subspace of $X^*$ generated by $x_1^*, \ldots, x_r^*$. It is easily seen that $X^* = Q_0 \oplus R$ and thus $X^*/Q_0$ has dimension $r$. But $X^* = Q_0 \oplus S_0$ where $S_0$ is $n$-dimensional, so $X^*/Q_0$ has dimension $n$. Hence $r = n$. 
(ii) Since $X^{**} = \pi(X) \oplus Q_0^+$, the result follows immediately from the proof of Theorem A.

(iii) This follows immediately from Theorem 7 of [3].

In view of 3.1, we adopt the following convention. When we say that $X$ is a quasi-reflexive Banach space, $X^* = Q \oplus R$ where $Q$ is total and $R$ is $n$-dimensional, we shall always mean that $X$ is to be considered in its equivalent norm so that its unit ball is compact in $\sigma(X, Q)$.

4. Induced topologies. In this section the topologies induced on $X$ by the decompositions $X^* = Q \oplus R$, $Q$ total, $R$ $n$-dimensional, are characterized as $w^*$-topologies.

4.1. Theorem. If $X$ is a quasi-reflexive Banach space, $X^* = Q \oplus R$ where $Q$ is total and $R$ is $n$-dimensional, then $X$ is equivalent to $Q^*$ under the mapping $v: X \rightarrow Q^*$ defined by $v(x)(x^*) = x^*(x)$, all $x^* \in Q$.

Proof. $v(x)$ is the contraction of $\pi(x)$ to $Q$. This is linear and 1-1, since $Q$ is total. By Theorem 9 of [2], $\pi(x)$ and its contraction to $Q$ have the same norm.

Hence $\sigma(X, Q)$ is merely the $w^*$-topology on $X$ when $X$ is considered as the conjugate space of $Q$ and properties which hold for general conjugate spaces thus hold for quasi-reflexive Banach spaces.

Bibliography


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