BASIC SEQUENCES IN THE SPACE OF MEASURABLE FUNCTIONS

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1. In a topological vector space $X$, a basic sequence $\{x_n\}$ is one whose finite linear combinations are dense in $X$. In a recent work, [1], A. A. Talalyan has observed that the space of measurable functions has a distinctly different character, with respect to the behavior of basic sequences, from, for example, the $L_p$ spaces, $p \geq 1$.

A striking result of Talalyan is the fact that if $\{\phi_n\}$ is basic, i.e., for every measurable $\phi$, there are finite linear combinations of the $\phi_n$ which converge almost everywhere to $\phi$, then if any finite number of functions is deleted from $\{\phi_n\}$, the remaining sequence is basic. This readily implies the existence of universal expansions, and the existence of a subsequence $\{\phi_{nk}\}$ which is basic even though the complement of the sequence $\{n_k\}$ is infinite.

The proof given by Talalyan necessitates the use of considerable machinery from the theory of orthonormal systems in $L_2$, and is quite involved. Our purpose is to show that the result follows almost immediately from the fact that the space $M$ of measurable functions, with the topology of convergence in measure, has a trivial dual.

2. With the topology of convergence in measure, the space $M$ of equivalence classes of measurable functions on $[0, 1]$ is a metrizable topological vector space. The following is well known, [2], but the proof is very short and is included for completeness.

**Lemma 1.** The dual $M'$ of $M$ consists only of 0.

**Proof.** Let $f$ be a continuous linear functional on $M$. If $f(x) \neq 0$ for some $x = x(t) \in M$, then, since the step functions are dense in $M$, there is a sequence $\{I_n\}$ of intervals, whose lengths converge to zero, such that $f(x_{I_n}) \neq 0$. There are then constants $k_n$ such that $f(k_n x_{I_n}) = 1$ for every $n$. But $\lim_n k_n x_{I_n} = 0$ in $M$.

We also need

**Lemma 2.** If the closure of the vector space generated by $\phi_1, \phi_2, \ldots$ is $M$, and the closure of the vector space generated by $\phi_2, \phi_3, \ldots$ is $X \subseteq M$, then $M$ is the span of $X$ and $\phi_1$.

**Proof.** If $\phi_1 \in X$, the result is obvious. Suppose $\phi_1 \notin X$. Let $x \in M$. There are $y_n \in X$, $a_n$ real, such that $\lim_n (a_n \phi_1 + y_n) = x$. If $\{a_n\}$ had a subsequence $\{a_{nk}\}$ converging to infinity, then...
\{\phi_1 + y_{nk}/a_{nk}\} \text{ would converge to zero, so that } \phi_1 = -\lim_{k} y_{nk}/a_{nk} \text{ would belong to } X. \text{ This being false, } \{a_n\} \text{ is bounded and has a convergent subsequence } \{a_{mk}\}. \text{ Let } a = \lim_{k} a_{mk}. \text{ Now } \lim_{k} (a_{mk}\phi_1 + y_{mk}) = x \text{ implies } \{y_{mk}\} \text{ converges to } y \in X, \text{ and thus } a\phi_1 + y = x.

**Corollary 1.** \(X = M\).

**Proof.** Suppose \(X \neq M\). In view of Lemma 2, there exists a continuous linear functional on \(M\) which vanishes on \(X\) and is 1 at \(\phi_1\). This contradicts Lemma 1.

3. Let \(\phi_1, \phi_2, \ldots, \phi_n, \ldots\) be a basic sequence in \(M\). By Corollary 1, the closure of the vector space generated by \(\phi_2, \phi_3, \ldots, \phi_n, \ldots\) is also \(M\). Indeed, for every \(k\), the sequence \(\phi_k, \phi_{k+1}, \ldots, \phi_n, \ldots\) is basic. We thus have the

**Theorem.** If \(\{\phi_n\}\) is a basic sequence in \(M\), it remains basic if any finite subset is deleted.

4. This theorem has two interesting corollaries. A series \(\sum_{n=1}^{\infty} a_n\phi_n\) is called universal in a space \(X\) if for every \(x \in X\), a subsequence of the sequence of partial sums of the series converges to \(x\).

**Corollary 2.** If \(\{\phi_n\}\) is a basic sequence in \(M\), there is a sequence \(\{a_n\}\) such that the series \(\sum_{n=1}^{\infty} a_n\phi_n\) is universal in \(M\).

**Proof.** The space \(M\) is metrizable with distance

\[d(x, y) = \int_0^1 \frac{|x(t) - y(t)|}{1 + |x(t) - y(t)|} \, dt.\]

Let \(\{x_n\}\) be a countable dense subset of \(M\). Choose a sequence of positive real numbers \(\{\epsilon_n\}\) which converges to zero. There exists a linear combination \(\sum_{n=1}^{\infty} a_n\phi_n\) such that \(d(x_1, \sum_{n=1}^{\infty} a_n\phi_n) < \epsilon_1\). In view of our theorem, there is a combination \(\sum_{k=1}^{b_1} a_n\phi_n\) which approximates \(x_2 - S_{b_1}\) to within \(\epsilon_2\). Similarly there is \(\sum_{k=b_1+1}^{b_2} a_n\phi_n\) approximating \(x_1 - S_{b_2}\) to within \(\epsilon_3\). We then determine \(\sum_{k=b_2+1}^{b_3} a_n\phi_n\) such that \(d(x_3 - S_{b_3}, \sum_{k=b_2+1}^{b_3} a_n\phi_n) < \epsilon_4\) and \(d(x_3 - S_{b_3}, \sum_{k=b_2+1}^{b_4} a_n\phi_n) < \epsilon_5\). We now begin again with \(x_1\) and continue this process indefinitely, approximating ever-lengthening chains. It is clear from the construction that for every \(x \in M\) there is a subsequence of the sequence of partial sums of \(\sum_{n=1}^{\infty} a_n\phi_n\) which converges to \(x\).

Another consequence of the theorem is

**Corollary 3.** If \(\{\phi_n\}\) is a basic sequence in \(M\), there is a subsequence obtained by an infinite number of deletions which is also basic.
PROOF. Consider a sequence \( \{e_n\} \) as described above. Delete \( \phi_1 \) from \( \{\phi_n\} \). \( \phi_1 \) can be approximated to within \( \varepsilon_1 \) by a combination of \( \phi_n \) with highest index \( n_2 \); delete \( \phi_{n+1} \). We note that \( \phi_1 \) and \( \phi_{n+1} \) can be approximated by the remaining functions to within \( \varepsilon_2 \). Let \( n_3 \) be the largest index used in these approximations. Clearly we may take \( n_3 > n_2 \). Delete \( \phi_{n+1} \). Now approximate the three functions deleted to within \( \varepsilon_3 \). As before, we take the largest index used, \( n_4 \), to be greater than \( n_3 \). Delete \( \phi_{n+1} \) and repeat this process. It is clear that the subsequence which remains is basic since the deleted elements are approximable by finite linear combinations of the remaining functions.

5. Remark. We note that the class \( M \) can be widened to include the extended real valued functions, i.e., those which take on infinite values on sets of positive measure.

This is clearly a consequence of an extension of almost everywhere convergence to include the case of sequences which diverge to \( +\infty \) or \( -\infty \) on sets of positive measure and the fact that infinite valued measurable functions are the limits, in this sense, of finite valued ones.

REFERENCES


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