A MATRIX INEQUALITY

H. O. CORDES

In his paper Beiträge zur Störungstheorie der Spektralzerlegung [2], E. Heinz has proved the following

**Theorem.** Let $A$ and $B$ be two self-adjoint positive operators of a Hilbert space $\mathcal{H}$ and let $Q$ be any arbitrary linear closed operator with the adjoint $Q^*$ defined in $\mathcal{H}$. Let $Q \subseteq \mathcal{A}$ and $Q^* \subseteq \mathcal{B}$ and let

\[
\|Qu\| \leq \|Au\|, \quad \text{for every } u \in \mathcal{Q},
\]
\[
\|Q^*u\| \leq \|Bu\|, \quad \text{for every } u \in \mathcal{Q}^*.
\]

**Statement:**

\[
|\langle Qu, v \rangle| \leq \|A^*u\| \|B^{1-v}\|, \quad \text{for every } u \in \mathcal{A}, v \in \mathcal{B},
\]

and for every $0 \leq v \leq 1$.

Although several other proofs and even generalizations of this remarkable and interesting estimate have been published by T. Kato [4], J. Dixmier [1] and E. Heinz [3], we are going to present here one more proof which shows the statement under a different aspect again and which perhaps has the advantage of using only very elementary arguments.

Our main tool will be the following very simple

**Lemma.** Let $T$ be a linear operator of an $n$-dimensional euclidean space and let $(u, v)$, $(u, v)_0$ be any two positive definite inner products defined for $u, v$ of this space. Let

\[
\|u\| = \{(u, u)^{1/2}, \quad \|u\|_0 = \{(u, u)_0^{1/2}
\]

and let

\[
\|T\| = \sup_{\|u\| = 1} \|Tu\|,
\]
\[
\|T\|_0 = \sup_{\|u\|_0 = 1} \|Tu\|_0.
\]

**Statement:** If $T$ is hermitian symmetric with respect to $(u, v)_0$, then

\[
\|T\|_0 \leq \|T\|.
\]

**Proof.** It is well known that

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\[ \|T\|_0 = \max_{i=1,\ldots,n} |\lambda_i| \]

where \(\lambda_i, i=1, \ldots, n\) are the \(n\) proper values of \(T\):

\[ T\phi_i = \lambda_i \phi_i, \quad (\phi_i, \phi_k)_0 = \delta_{ik}, \quad i, k = 1, \ldots, n. \]

But for each \(\lambda_i\) we obtain

\[ \|T\phi_i\| = |\lambda_i| \|\phi_i\| \leq \|T\| \|\phi_i\|. \]

Since by assumption both inner products are positive definite it follows that

\[ |\lambda_i| \leq \|T\|, \quad i = 1, \ldots, n. \]

Hence

\[ \|T_0\| = \max |\lambda_i| \leq \|T\| \]

follows immediately.

In order to use the above lemma for the proof of our theorem we first restrict ourselves to the case of a finite dimensional Hilbert space \(\mathcal{H}\) and also we impose the further assumption that \(A\) and \(B\) both have a bounded inverse. These additional assumptions will be removed later.

By substitutions of the form \(Au = v\), resp. \(Bu = v\) the two inequalities

\[ \|Qu\| \leq \|Au\|, \quad \|Q^*u\| \leq \|Bu\| \]

go into

\[ \|QA^{-1}v\| \leq \|v\|, \quad \|Q^*B^{-1}v\| \leq \|v\|. \]

Since the adjoint of every bounded operator is bounded by the same constant as the original operator, it follows that

\[ \|A^{-1}Q^*u\| \leq \|u\|, \quad \|B^{-1}Q^*u\| \leq \|u\|. \]

Consequently

\[ \|A^{-1}Q^*B^{-1}Q\| \leq 1. \]

Since the operator \(T = A^{-1}Q^*B^{-1}Q\) is hermitian with respect to the positive definite inner product

\[ (u, v)_0 = (u, Av), \]

we get

\[ \|T\|_0 \leq \|T\| \leq 1. \]

Now
follows. Introducing again the inner product \((u, v)\) we get 
\[
(Qu, B^{-1}Qu) \leq (u, Au)
\]
or
\[
\|B^{-1/2}QA^{-1/2}\| \leq 2 \|v\|^2.
\]
Finally
\[
| (B^{-1/2}QA^{-1/2}u, v) | \leq \|u\| \|v\|
\]
or
\[
| (Qu, v) | \leq \|A^{1/2}u\| \|B^{1/2}v\|
\]
follows.

This is the statement for \(v = 1/2\).

In order to prove the theorem for the general case we use an argument which looks similar to that used by T. Kato [4], but it does not seem to be the same.

We assume the statement to be true for all \(v = m/2^k, m = 0, 1, 2, \ldots, 2^k\) and we show that from there it follows by induction for every \(v = m/2^{k+1}, m = 0, 1, 2, \ldots, 2^{k+1}\).

Since by the above arguments the statement has been shown to be true for \(v = 0, 1/2, 1\), that means for \(k = 1\), this amounts to a proof for every number of the type \(k/2^m, k, m\) being arbitrary.

From
\[
| (Qu, v) | \leq \|A^{m/2^k}u\| \|B^{1-m/2^k}v\|
\]
we conclude that
\[
| (QA^{-m/2^k}u, v) | \leq \|u\| \|uB^{1-m/2^k}v\|.
\]
This amounts to
\[
\|A^{-m/2^k}Qv\| \leq \|B^{1-m/2^k}v\|.
\]

On the other hand
\[
\|QA^{-m/2^k}u\| \leq \|A^{1-m/2^k}u\|
\]
follows from our initial assumption
\[
\|Qu\| \leq \|Au\|.
\]
Now let
\[
Q_1 = QA^{-m/2^k}, \quad A_1 = A^{1-m/2^k}, \quad B_1 = B^{1-m/2^k}.
\]
Then the above inequalities amount to
\[ \| Q v \| \leq \| A v \|, \quad \| Q^* u \| \leq \| B v \|. \]
By application of the theorem for \( \nu = 1/2 \) we get
\[ \| (Q A^{-m/2^k} u, v) \| \leq \| A^{1/2(1-m/2^k)} u \| \| B^{1/2(1-m/2^k)} v \|, \]
or
\[ \| (Q u, v) \| \leq \| A^{1/2+m/2^k+1} u \| \| B^{1/2-m/2^k+1} v \|. \]
Hence the assertion follows for
\[ \nu = \frac{2^k + m}{2^{k+1}}, \quad m = 0, 1, 2, \ldots, 2^k \]
and by reasons of symmetry for
\[ \nu = \frac{2^k - m}{2^{k+1}}, \quad m = 0, 1, 2, \ldots, 2^k \text{ too.} \]
Together we conclude the assertion for all
\[ \nu = \frac{m}{2^{k+1}}, \quad m = 0, 1, \ldots, 2^{k+1}. \]
Since finally \( A^r \) and \( B^{1-r} \) depend continuously on \( \nu \) and since the set of all numbers of the form \( m/2^k \) is dense in the interval \( 0 \leq \nu \leq 1 \) we get the assertion for every \( \nu \) of this interval.
From the continuity of \( A^r \) and \( B^{1-r} \) in \( A \) and \( B \) follows further that the assumption of existence of \( A^{-1} \) and \( B^{-1} \) can be removed. For if \( A^{-1} B^{-1} \) does not exist, then replace \( A \) and \( B \) by \( A + \epsilon, B + \epsilon \), \( \epsilon > 0 \), respectively. Because of
\[ \| A u \| \leq \| (A + \epsilon) u \| \quad \text{and} \quad \| B u \| \leq \| (B + \epsilon) u \| \]
\( (A \text{ and } B \text{ are positive definite}) \) we conclude
\[ \| Q u \| \leq \| (A + \epsilon) u \|, \quad \| Q^* u \| \leq \| (B + \epsilon) u \|. \]
Now \( A + \epsilon, B + \epsilon \) have an inverse, hence
\[ \| (Q u, v) \| \leq \| (A + \epsilon)^r u \| \| (B + \epsilon)^{1-r} u \| \]
for every \( \epsilon \). If \( \epsilon \) tends to zero it follows
\[ \| (Q u, v) \| \leq \| A^r u \| \| B^{1-r} v \|. \]
for the more general class of $A$, $B$ too.

Finally every conclusion except the proof of the lemma works for
an infinitely dimensional Hilbert space too provided that all oper-
ators are assumed to be bounded. The lemma can easily be proved
for this more general case too under the additional assumption that
the two metrics $\|u\|$ and $\|u\|_0$ be topologically equivalent:

$$c\|u\|_0 \leq \|u\| \leq C\|u\|_0.$$ 

The proof can be given as follows:

If $\|u\|_0$ and $\|u\|$ are equivalent norms then an operator is bounded
in both norms if it is bounded in one of the two norms only. Hence
the spectrum $\sigma$ of an operator $T$ (i.e. the set of $\lambda$ for which $(T-\lambda)^{-1}$
is bounded) is the same under both norms.

Let

$$\|\|T\|\| = \sup_{\lambda \in \sigma} |\lambda|$$

be the spectral norm of $T$.

Then, as is well known,

$$\|\|T\|\| = \|T\|_0,$$

if $T$ is hermitian symmetric (or even only normal) under $(u, v)_0$ and

$$\|\|T\|\| \leq \|T\|.$$

Hence

$$\|T\|_0 \leq \|T\|$$

and the lemma is proven.

Finally the extension of the theorem to the case of general un-
bounded operators follows in the same manner as in [2].

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UNIVERSITY OF CALIFORNIA, BERKELEY