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ISOTOPY IN 3-MANIFOLDS. III. CONNECTIVITY OF SPACES OF HOMEOMORPHISMS

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1. **Introduction.** J. H. Roberts has reported a proof that the space $H(P)$ of homeomorphisms of the plane onto itself has exactly two components [**7**]. The corresponding result for three-space is proved in this paper (Roberts stated in an indirect communication to the author that he has carried out investigations along this line but does not intend to publish them). M. K. Fort proved that $H(P)$ is locally arcwise connected [**4**] and Fort and E. E. Floyd published a paper in which they proved that the space of homeomorphisms of the 2-sphere onto itself is uniformly locally connected [**3**]. Both of these results are likewise extended in this paper to the three-dimensional analogues.

2. **Definitions.** All of the above-mentioned results are for the compact-open topology introduced by R. H. Fox [**5**]. If $d(x, y)$ denotes the usual metric for Euclidean 3-space, E^3, and C_r is the cube in E^3 with center at the origin and side r (faces parallel to a fixed set of coordinate planes), then an admissible metric for the space, $H(E)$, of homeomorphisms of E^3 onto itself under the compact-open topology is $\rho(f, g) = \inf_{r>0} \max[1/r, \sup_{x \in C_r} d(f(x), g(x))]$. Similarly, an admissible metric for the space, $H(S)$, of homeomorphisms of the 3-sphere, S^3, onto itself is $\rho(f, g) = \sup_{x \in S^3} d(f(x), g(x))$. Since the treat-

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ment of E^3 and S^3 will be separate, $d(x, y)$ will be used for the usual metric in S^3 as well as E^3 and $\rho(f, g)$ for the metric in $H(S)$ as well as $H(E)$.

An isotopy between homeomorphisms f and g, defined on a space X into a space Y, is a (continuous) mapping G of the cartesian product, $X \times I$, of X and the closed unit interval, I, into Y such that $G(x, t)$ is a homeomorphism for each fixed $t \in I$, and for all $x \in X$, $G(x, 0) = f(x)$ and $G(x, 1) = g(x)$. G is an ϵ-isotopy on a subset A of X if for every point $p \in A$, the image of $p \times I$ under G has diameter less than ϵ. If $A = X$, G is simply called an ϵ-isotopy.

If $h \in H(E)$ and $f \in H(E)$, f is said to be a δ-approximation to h on the subset A of E^3 provided $d(f(x), h(x)) < \delta$ for all $x \in A$. If $A = E^3$, f is referred to as simply a δ-approximation.

In this paper, a homeomorphism h of E^3 onto itself will be called piecewise linear if it is linear on every simplex of some rectilinear triangulation of E^3. An isotopy G is piecewise linear if $G(x, t)$ is piecewise linear for each $t \in I$.

3. **Preliminary lemmas.** The first lemma is stated without proof since it is a direct result of Theorem 1 of [**8**]. Lemma 2 is an adaptation of a result given by Alexander [**1**] and is proved for the sake of completeness even though it is a direct consequence of Alexander's work.

LEMMA 1. *Given $\epsilon > 0$ and a homeomorphism h of E^3 onto itself, suppose C is a cube in E^3 and U is any open set containing the boundary,* $\mathrm{Bd}(C)$, *of C. Then there is a $\delta > 0$ such that if f and g are piecewise linear δ-approximations to h on U, a piecewise linear ϵ-isotopy G exists such that* (i) $G(x, 0) = f(x)$ *for* $x \in E^3$, (ii) $G(x, 1) = g(x)$ *for* $x \in \mathrm{Bd}(C)$, *and* (iii) $G(x, t) = f(x)$ *for* $x \in E^3 - U$ *for all* $t \in I$.

Once having "fitted" f to g along $\mathrm{Bd}(C)$ by Lemma 1, it is a simple matter to deform f slightly in a piecewise linear manner so that $f = g$ on the center of C as well. Then Alexander's result can be applied to fit f onto g on all of C as indicated in the following lemma.

LEMMA 2. *Let ϵ, h, and C be as in Lemma* 1 *and suppose f and g are piecewise linear δ-approximations to h on C with $f = g$ on* $\mathrm{Bd}(C)$. *Then if δ is sufficiently small, there is a piecewise linear ϵ-isotopy F such that* (i) $F(x, 0) = f(x)$ *for* $x \in E^3$, (ii) $F(x, 1) = g(x)$ *for* $x \in C$, *and* (iii) $F(x, t) = f(x)$ *for* $x \in \mathrm{Cl}[E^3 - C]$ *for all* $t \in I$.

PROOF. We assume, without loss of generality, that $f = g$ on the center of C as well. To apply Alexander's method, let ϕ be a γ-approximation to the identity on C which is the identity on $\mathrm{Cl}[E^3-C]$ and the center of C. If x_i $(i=1, 2, 3)$ are the coordinates of any point referred to the coordinate system in which $C=C_1$, let x_i' denote the coordinates of its image under ϕ. Define an isotopy G as follows: $G(p, 0)=p$, and if $0<t\leqq 1$ and p has coordinates tx_i, $G(p, t)=q$ where q has coordinates tx_i'. Thus $G(p, t)$ is the identity outside C_t and maps C_t onto itself exactly like ϕ maps C onto itself. Thus it is easily seen that G is a 2γ-isotopy of the identity onto ϕ which is fixed outside C and is piecewise linear if ϕ is. Denote the identity by i.

To prove the lemma, suppose $0<\gamma<\epsilon/4$ and choose γ' so that for any subset A of C having diameter less than γ', $h^{-1}(A)$ has diameter less than γ. Choosing $\delta<\min(\gamma'/2, \epsilon/4)$, it follows that $f^{-1}g$ is a γ-approximation to the identity on C. To see this, note that since $g(x)=f(f^{-1}g(x))$, and f is a δ-approximation to h, it follows that $d(g(x), hf^{-1}g(x))<\delta$ for all $x\in C$. Hence, by the triangle inequality, $d(h(x), hf^{-1}g(x))<2\delta<\gamma'$ and therefore $d(x, f^{-1}g(x))<\gamma$ by the choice of γ'. Let G be the 2γ-isotopy between i and $f^{-1}g$ obtained by Alexander's method. Then since δ and γ are both less than $\epsilon/4$, $F=fG$ is a piecewise linear ϵ-isotopy which satisfies (i)–(iii).

4. The principal results.

THEOREM 1. *Given a positive number ϵ, every homeomorphism h of E^3 onto itself is ϵ-isotopic to a piecewise linear homeomorphism.*

PROOF. Consider the cubes C_n in E^3, and choose a decreasing sequence of positive numbers $\{\delta_n\}$ such that δ_n satisfies Lemma 2 for $\epsilon/2^n$, h, and C_n in place of ϵ, h, and C. Next, given a sequence of disjoint open sets $\{U_n\}$ where $U_n\supset \mathrm{Bd}(C_n)$, define a decreasing sequence of positive numbers $\{\delta_n'\}$ so that δ_n' is less than $\delta_n/2$ and satisfies Lemma 1 for $\delta_n/2$, h, C_n, and U_n in place of ϵ, h, C, and U.

Moise has proved that there exist arbitrarily close piecewise linear approximations to h [6, Theorem 2]; therefore choose a sequence $\{f_n'\}$ of δ_n'-approximations. Then for each n, a piecewise linear $\delta_n/2$-isotopy G_n can be defined which deforms f_n' so as to agree with f_{n+1}' on $\mathrm{Bd}(C_n)$ and is the identity outside U_n. This replaces $\{f_n'\}$ by a sequence $\{f_n\}$ of piecewise linear δ_n-approximations to h, where $f_n(x)=G_n(f_n'(x), 1)$.

Now let f be the piecewise linear homeomorphism which is equal to f_n on C_n-C_{n-1} for $n=1, 2, 3, \cdots$ (let C_0 be the null set). Both f and f_2 are δ_1-approximations to h on C_1, so by choice of δ_1 there is a piecewise linear $\epsilon/2$-isotopy F_1 which fits f to f_2 on C_1 and is the identity outside C_1. Now $F_1(f(x), 1)$ agrees with f_2 on all of C_2. For

$n>1$, if f' is the result of the first $n-1$ such isotopies, then f' and f_{n+1} are δ_n-approximations to h on C_n, hence an $\epsilon/2^n$-isotopy F_n exists which fits f' to f_{n+1} on C_n so that it agrees with f_{n+1} on all of C_{n+1}.

Define F to be the product of all the F_n. That is, F is the isotopy which carries out the entire deformation of F_1 in the interval $0\leqq t \leqq 1/2$, that of F_2 in $1/2\leqq t\leqq 3/4$, F_3 in $3/4\leqq t\leqq 7/8$, etc., and $F(x, 1) = h(x)$. Then F is an ϵ-isotopy between h and the piecewise linear homeomorphism f as required by the theorem. Note that F is piecewise linear for $0\leqq t<1$.

THEOREM 2. *The space $H(E)$ of homeomorphisms of E^3 onto itself under the compact-open topology is locally arcwise connected.*

PROOF. Note that two homeomorphisms are joined by an arc of diameter less than ϵ in the compact-open topology provided there is an isotopy between them which is an ϵ-isotopy on $C_{1/\epsilon}$. A simple modification of Alexander's method extends the isotopy of Lemma 2 to such an isotopy between f and g.

Suppose now $h\in H(E)$ and U is an ϵ-neighborhood of h. By Lemmas 1 and 2 and the above remarks, there is a positive number $\delta<\epsilon/3$ such that any two piecewise linear δ-approximations to h on $C_{1/\epsilon}$ are joined by an arc of diameter less than $\epsilon/3$. Let V be a δ-neighborhood of h. Then $h'\in V$ implies that $\sup_{x\in C_{1/\delta}} d(h(x), h'(x)) =\gamma<\delta$. Now by Theorem 1, both h and h' are $(\delta-\gamma)$-isotopic (and *a fortiori* $\epsilon/3$-isotopic) to piecewise linear homeomorphisms f and g, respectively. Then, since $C_{1/\delta}\supset C_{1/\epsilon}$, f and g are δ-approximations to h on $C_{1/\epsilon}$ and so can be joined by an arc of diameter less than $\epsilon/3$. This proves the existence of an arc of diameter less than ϵ joining h and h'. Thus $H(E)$ is locally arcwise connected.

THEOREM 3. *$H(E)$ consists of exactly two components.*

PROOF. Isotopy preserves the Brouwer degree of a homeomorphism of E^3 onto itself (see Chapter 12 of [2] and also [9]). That is to say, if $h\in H(E)$ is orientation preserving, so also is any member of $H(E)$ isotopic to h. Then the identity and the reflection in the origin are not isotopic since they have Brouwer degree 1 and -1, respectively. Every member of $H(E)$ has Brouwer degree 1 or -1, and Theorem 2 implies these two disjoint subsets of $H(E)$ are open. Hence there are at least two components in $H(E)$. That there are exactly two follows easily from Theorem 1.

Suppose f and g are two elements of $H(E)$ having the same Brouwer degree. Then by Theorem 1 they are isotopic to piecewise linear homeomorphisms f' and g', respectively, which must be of the same degree. Let T be a 3-simplex on which both f' and g' are linear. Since $f'(T)$ and $g'(T)$ have the same orientation, there is an isotopy which deforms f' so as to agree with g' on T. This isotopy can then be extended to an isotopy between f' and g' by an application of Alexander's method to the exterior of T. Thus f and g are isotopic and therefore belong to the same component of $H(E)$. This completes the proof of the theorem.

THEOREM 4. *The space $H(S)$ of homeomorphisms of the 3-sphere, S^3, onto itself under the compact-open topology is uniformly locally arcwise connected.*

PROOF. Since $\rho(f, g) = \rho(fh, gh)$ in $H(S)$, it suffices to prove that $H(S)$ is locally arcwise connected at the identity, i.

Let p be the point $(0, 0, 0, 1)$ of S^3 and $F(S)$ the subspace of $H(S)$ consisting of all elements f for which $f(p) = p$. If π is the stereographic projection of $S^3 - p$ from p onto E^3, it is easily shown that $\pi f \pi^{-1}$ defines a homeomorphism of $F(S)$ onto $H(E)$. Then since $H(E)$ is locally arcwise connected, so also is $F(S)$.

Thus, given $\epsilon > 0$, choose $\delta < \epsilon/2$ such that for every f of $F(S)$ in the 2δ-neighborhood of i there is an $\epsilon/2$-isotopy between f and i. Now, given any h of $H(S)$ in the δ-neighborhood of i, $d(h(p), p) < \delta < \epsilon/2$, therefore there is a rotation of S^3 defining a δ-isotopy (which is also an $\epsilon/2$-isotopy) of h onto an element f of $F(S)$. Since f is within 2δ of i, by hypothesis on δ there exists an $\epsilon/2$-isotopy of f onto i. Thus h is ϵ-isotopic to i. This completes the proof.

COROLLARY. *Given a positive ϵ, every homeomorphism of S^3 onto itself is ϵ-isotopic to a piecewise linear homeomorphism.*

REMARKS. If it were true that every monotone mapping of S^3 onto itself is a limit point of $H(S)$, then the characterization of monotone mappings given by Floyd and Fort for the 2-sphere [3] could be extended to the 3-sphere. Namely, that if Q is the interior of S^3, a mapping f of S^3 onto itself is monotone if and only if there exists a continuous extension g of f such that g maps $S^3 \cup Q$ onto itself and $g|Q$ is a homeomorphism of Q onto Q. J. W. T. Youngs [10] proved the required approximation theorem for monotone mappings of S^2 onto itself but so far as the author knows, this has not been extended to S^3. On p. 96 of [11], Floyd asks what maps of S^3 onto itself can be so approximated.

It appears that the methods used here can be extended to more general 3-manifolds. J. H. Roberts, M. E. Hamstrom, and the author have obtained some results along this line but none have been published as yet.

Added in proof. The author reported a proof of the existence, for any $\epsilon > 0$, of an ϵ-isotopy between homeomorphisms f and g of E^3 if $d(f(x), g(x)) < \delta$ for all $x \in E^3$, provided δ is small enough (Bull. Amer. Math. Soc. Abstract 63-4-546). Since this paper was submitted, J. M. Kister has improved and extended this result to E^n and has obtained other related results (see James Kister, *Small isotopies in Euclidean spaces and 3-manifolds*, Bull. Amer. Math. Soc. vol. 65 (1959) pp. 371–373).

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