A SUFFICIENT CONDITION THAT A MAPPING OF
RIEMANNIAN MANIFOLDS BE A FIBRE BUNDLE

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1. Introduction. All manifolds, maps, tensor-fields, curves, etc. will be of differentiability class \( C^\infty \) unless mentioned otherwise. If \( x \) is a point of a manifold \( X \), \( X_x \) denotes the tangent space at \( x \). If \( \phi: X \to B \) is a \( C^\infty \) map of manifolds, \( \phi_*: X_x \to B_{\phi(x)} \) is the linear map on tangent vectors induced by \( \phi \). We suppose

(a) \( \phi \) is of maximal rank on \( X \), i.e. \( \phi_* (X_x) = B_{\phi(x)} \) for all \( x \in X \).

(b) \( X \) and \( B \) are Riemannian manifolds, and the isomorphism

\[
\phi_*: X_x/\phi_*^{-1}(0) \to B_{\phi(x)}
\]

preserves the inner-products defined by the metrics on these spaces, for all \( x \) in \( X \).

The main theorem is:

**Theorem 1.** If \( X \) is complete as a Riemannian space, so is \( B \). \( \phi \) is then a locally trivial fibre space. If in addition the fibres of \( \phi \) are totally geodesic submanifolds of \( X \), \( \phi \) is a fibre bundle with structure group the Lie group of isometries of the fibre.

2. Generalities [2; 1]. If \( x \in X \), \( v_1, v_2 \in X_x \), let \( (v_1, v_2) \) denote the inner product that defines the Riemannian metric. If \( \sigma: [0, 1] \to X \) is a curve, let \( \sigma': t \to \sigma'(t) \) denote the tangent vector field to \( \sigma \). If \( v: t \to v(t) \in X_{\sigma(t)} \) is a vector-field along \( \sigma \), let \( \Delta v \) denote the covariant derivative of \( v \) along \( \sigma \) [1].

Let us recall how it is defined by Elie Cartan’s method of orthonormal moving frames: Suppose \( U \) is an open set of \( X \) and \( w_i \) \((1 \leq i, j, k \cdots \leq n = \dim X\), summation convention\) \( 1 \)-differential forms in \( U \) with

\[
ds^2 = w_i \cdot w_i \quad \text{(Symmetric Product)}.
\]

There are unique \( 1 \)-forms \( w_{ij} \) in \( U \), called the connection forms, such that

\[
dw_i = w_{ij} \wedge w_j \quad \text{(Exterior Product)},
\]

\[
w_{ij} = -w_{ji}.
\]

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Then, for the portion of $\sigma$ in $U$, $\Delta v(t) \in X_{\sigma(t)}$ is defined by the condition:

$$w_i(\Delta v(t)) = \frac{d}{dt} w_i(v(t)) - w_{ij}(\sigma'(t))w_j(v(t)).$$

Suppose now that $\sigma_*$ is a one-parameter family of curves, a deformation of $\sigma$, i.e. $\sigma_0 = \sigma$, and $v$ is the corresponding infinitesimal deformation field, i.e. $v(t)$ is the tangent vector to the curve $s \rightarrow \sigma_s(t)$ at $s = 0$. Put $f(s) = \text{length of the curve } \sigma_s$, and suppose each curve $\sigma_s$ is parametrized proportionally to arc-length. The formula for the "first variation" of arc length is then:

$$f'(0) = \int_0^1 (\sigma'(t), \Delta v(t))/f(0) dt$$

$$= (\sigma'(t), v(t))/f(0) \bigg|_{t=0}^t - \int_0^1 (\Delta \sigma'(t), v(t))/f(0) dt.$$

The condition that $\sigma$ be a geodesic is then

$$\Delta \sigma'(t) = 0.$$

3. Proof of Theorem 1. Let us return now to the map $\phi: X \rightarrow B$ of §1. For $x \in X$, a $v \in X_x$ is said to be vertical if $\phi_*(v) = 0$. The vectors in the orthogonal complement to the vertical vectors are called horizontal. $P_V$ and $P_H$ denote the projections of $X_x$ on these subspaces. The field $x \rightarrow P_V(X_x)$ of tangent subspaces defines, with the metric, a "bundle-like" foliation on $X$ [7]. A curve in $X$ is said to be horizontal if its tangent vector field is horizontal. If $\sigma: [0, 1] \rightarrow B$ is a curve, $x_0 \in \phi^{-1}(\sigma(0))$, there is at most one horizontal lifting $\sigma_1: [0, 1] \rightarrow X$ of $\sigma$ beginning at $x_0$, and horizontal liftings always exist locally [4]. If they always exist globally as well, the field of horizontal subspaces is a connection for the map $\phi$ in the sense of Ehresmann [4], and the map $\phi$ is a locally trivial fibre space.

**Proposition 3.1.** If $\sigma: [0, 1] \rightarrow X$ is a curve, $v$ a vector field along $\sigma$, $\phi_\sigma: t \rightarrow \phi_\sigma(t)$ and $\phi_*v: t \rightarrow \phi_*v(t)$ the projections in $B$, then

$$\Delta \phi_*(v) = \phi_*(\Delta(P_H(v))).$$

Hence in particular the projection of a horizontal geodesic of $X$ is a geodesic of $B$, and conversely the horizontal lifting of a geodesic of $B$, if it exists, is a geodesic.

**Proof.** Suppose $\dim B = n - m$. Adopt the following ranges of in-
indices and summation conventions: \(1 \leq a, b, c, \leq m; m+1 \leq u, v, \cdots \leq n\). Let \(\theta_u\) be 1-forms in an open set \(U\) of \(B\) with
\[
d s^2 = \theta_u \cdot \theta_u.
\]

Suppose \(\theta_{uv}\) are the corresponding connection forms. Let \(w_i\) be 1-forms in an open set \(U'\) of \(X\) with \(\phi(U') = U\) and \(ds^2 = w_i \cdot w_i\), \(w_u = \) the one form on \(U'\) induced by \(\phi\) from \(\theta_u\), to be denoted here by \(\phi^*(\theta_u)\).

Suppose \(\theta_{uv} = f_{uvw} \theta_w, w_{ij} = g_{ijk} w_k\), where \(w_{ij}\) are the connection forms corresponding to \(w_i\) in \(U'\). One derives by calculation:

\[
\begin{align*}
(a) & \quad g_{uvw} = \phi^*(f_{uvw}), \\
(b) & \quad g_{uvu} = g_{uva}, \\
(c) & \quad g_{uab} = g_{uba}.
\end{align*}
\]

(3.2a) now implies 3.1, via 2.1.

**Proposition 3.2.** If \(X\) is complete as a Riemannian metric space, so is \(B\). In this case the horizontal liftings of paths\(^2\) of \(B\) exist globally, i.e. \(\phi\) is a fibre space.

**Proof.** The horizontal lifting of a small geodesic segment of \(B\) is a geodesic segment. If \(X\) is complete it can be completely extended to a geodesic of infinite length. The projection in \(B\) of the extension is an extension of the given segment to an infinite geodesic, hence every geodesic segment of \(B\) can be indefinitely extended.

To prove the second part, note that the horizontal lifting of a path of \(B\) has the same length. If \(\sigma: [0, 1] \to B\) is a path, local horizontal liftings always exist; if we try to get a global lifting by the usual process of continuation we run into no obstruction since the partial liftings will always lie in a fixed bounded, hence compact region of \(X\).

We assume from now on that every path in \(B\) has horizontal liftings starting at arbitrary points of the initial fibre. If \(\sigma\) is a path: \([0, 1] \to B\), there is a diffeomorphism \(h_\sigma: \phi^{-1}(\sigma(0)) \to \phi^{-1}(\sigma(1))\) obtained by mapping each \(x_0 \in \phi^{-1}(\sigma(0))\) into the end-point of the horizontal lifting of \(\sigma\) starting at \(x_0\) \([4]\).

**Proposition 3.3.** If all the fibres of \(\phi\) are totally geodesic submanifolds of \(X\), then for each path \(\sigma\) of \(B\), \(h_\sigma\) is an isometry of \(\phi^{-1}(\sigma(0))\) onto \(\phi^{-1}(\sigma(1))\).

**Proof.** We use the first variation formula 2.2. Suppose \(\alpha: [0, 1] \to X\)

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\(^2\) A path is a continuous image of \([0, 1]\) that is composed of a finite number of unbroken \(C^\infty\) curves.
is a curve completely on a fibre of \( \phi \), i.e. a vertical curve. Suppose we deform \( \alpha \) in a horizontal direction i.e. \( \alpha_s \) is a one-parameter family of curves, with infinitesimal deformation field \( v: t \to v(t) \in X_{\alpha(t)} \) a horizontal curve. If \( f(s) = \text{length of } \alpha_s \), 2.2 says that \( f'(0) = 0 \), since \( \Delta \alpha' \) is a vertical vector field (because the fibre is totally geodesic).

If \( \sigma: [0, 1] \to B \) is a curve, the image curve of a curve in \( \phi^{-1}(\sigma(0)) \) by \( h \) is obtained by deforming along horizontal curves covering \( \sigma \), hence has the same length. Q.E.D.

We assume then that the fibres of \( \phi \) are totally geodesic. Fix a \( b_0 \in B \), and set \( F = \phi^{-1}(b_0) \). Let \( G \) be the Lie group of isometries of \( F \). If \( b \in B \), let \( G_b \) be the set of isometries of \( F \) onto \( \phi^{-1}(b) \). Let \( E \) be the point-set union of all \( G_b \), for \( b \in B \). Let \( \bar{p}: E \to B \) be the map \( G_b \to b \). \( G \) acts in an obvious way on \( E \) on the left, and \( E \) would be a principal fibre bundle with \( G \) as structure group, except that so far it has no topology or manifold structure. Suppose though one can find a locally finite open covering of \( B \), \( \{ U \} \), such that in each open set \( U \) of the covering there is a cross-section \( f_U : U \to E \). Then, for each \( U \) and \( U' \) of the covering that intersect, there is a map \( g_{UU'} : U \cap U' \to G \) such that

\[
f_{U'}(b) = g_{UU'}(b)f_U(b) \quad \text{for } b \in U \cap U'.
\]

Suppose that the covering and the cross-sections can be chosen so that all the maps \( g_{UU'} \) are \( C^\infty \); clearly then they define a \( C^\infty \) coordinate bundle in the sense of Steenrod [8] which is isomorphic to \( E \), and one can use this isomorphism to define \( E \) as a \( C^\infty \) manifold.

We use Proposition 3.3 to define such cross-sections. Choose the covering so that each \( U \) is a geodesically convex open set of \( B \), and pick a point \( b_U \) in \( U \) and a curve \( c_U \) joining \( b_0 \) to each \( b_U \). Any \( b \in U \) is on a unique geodesic with \( b_U \), and one obtains a path \( \sigma_{b,U} \) depending on \( b \) and \( U \) joining \( b_0 \) with \( b \). Define \( f_U(b) = h_{\sigma_{b,U}B} \in B_b \). It is easy to see that the transition functions \( g_{UU'} : G \) are \( C^\infty \); they are defined by solutions of \( C^\infty \) ordinary differential equations depending in a \( C^\infty \) way on initial conditions.

Having defined \( E \) as a \( C^\infty \) principal fibre bundle with structure group \( G \), it is clear that the given fibre space \( \phi: X \to B \) is isomorphic to the associated bundle with fibre \( F \). The proof of Theorem 1 is then finished.

Let \( \Omega \) be the space of loops of \( B \) starting at \( b_0 \). \( h: \sigma \to h_{\sigma} \) is then a map \( \Omega \to G \). Let \( B_G \) be the base of the universal classifying bundle for \( G \). The bundle \( E \) is induced by a map, the classifying map, \( B \to B_G \). The loops of \( B_G \) have the same homotopy type as \( G \), and it is clear that, up to homotopy, \( h \) is the same as the map \( \Omega \to \Omega(B_G) \) induced by the classifying map [6].
4. Some examples. First, note that the homogeneous spaces of Lie groups with compact isotropy groups provide a class of examples satisfying the hypotheses of Theorem 1. Of course, Theorem 1 is well-known for these examples [8]. This suggests as generalization the mappings \( \phi: X \to B \) which locally have the structure of principal fibre bundle with a Cartan connection [5]. Recall that the bundle of orthonormal frames of a Riemannian manifold has this sort of structure.

To describe these examples, one should be given

1) a map \( \phi: X \to B \) of manifolds which is of maximal rank everywhere, with \( \dim X = n, \dim B = n - m \),

2) a real Lie algebra of dimension \( n \), \( G \), a subalgebra \( K \) of dimension \( m \), a subspace \( M \) of \( G \) with \( G = K \oplus M \), \( [K, M] \subset M \), a positive definite bilinear form on \( M \) with respect to which the operations of \( \text{Ad } K \) are skew-symmetric. (We follow Chevalley [3] for the notations of Lie algebra theory.) Choose a basis \( \mathfrak{g}_i \) of \( G \) such that (a) the \( \mathfrak{g}_a \) span \( K \) and the \( \mathfrak{g}_u \) span \( M \) (same range of indices as in §3), (b) the \( \mathfrak{g}_u \) are an orthonormal basis with respect to the bilinear form on \( M \). Let \( [\mathfrak{g}_i, \mathfrak{g}_j] = c_{ijk} \mathfrak{g}_k \) be the structural equations of \( G \) with respect to this basis.

3) \( n \) vector-fields \( \theta_i \) for \( X \) which are linearly independent at every point of \( X \), i.e. form a parallelism for \( X \), such that

(a) \( [\theta_a, \theta_i] = c_{aij} \theta_j \), i.e. only part of the structural equations are satisfied, and

(b) for \( x \in X \), the \( \theta_a(x) \) form a basis for the vectors tangent to the fibre of \( \phi \) through \( x \).

One then defines a Riemannian metric for \( X \) by the requirement that, for \( x \in X \), the \( \theta_i(x) \) form an orthonormal basis for \( X_x \), i.e. if \( w_i \) are the dual basis of differential forms of \( X \), \( w_i(\theta_j) = \delta_{ij} \), then

\[
d s^2 = w_i \cdot w_i.
\]

Call the vectors spanned by the \( \theta_a \) the vertical vectors and let \( P_H \) and \( P_V \) be the projections defined on \( X_x \), as before.

**Proposition 4.1.** With this metric for \( X \), if the fibres of \( \phi \) are connected \( B \) inherits a Riemannian metric such that, for each \( x \), the isomorphism \( \phi_*: P_H(X_x) \to B_{\phi(x)} \) preserves inner products and the fibres of \( \phi \) are totally geodesic.

**Proof.** Suppose \( [\theta_i, \theta_j] = f_{ijk} \theta_k \), where the \( f_{ijk} \) are real-valued \( C^\infty \) functions on \( X \), \( f_{ijk} = c_{ijk} \). Then [3, p. 154]

\[
(4.1) \quad dw_i = \frac{1}{2} f_{jk} w_j \wedge w_k.
\]
Suppose $w_{ij} = h_{ijk}w_k$ are the connection forms for the Riemannian metric, i.e. $dw_i = w_{ij} \wedge w_j$, $w_{ij} = -w_{ji}$.

One calculates:

\[(4.2)\]
\[h_{ijk} = \frac{1}{2} (f_{kji} - f_{jik} + f_{ikj})\]

Then, if $\sigma: [0, 1] \rightarrow K$ is a curve, and $v: t \rightarrow v(t) \in X_{\sigma(t)}$ is a vector field along $\sigma$,

\[(4.3)\]
\[w_i(\Delta v(t)) = \frac{d}{dt} w_i(v(t)) - h_{ijk}(\sigma'(t))w_k(\sigma'(t))w_j(v(t)).\]

Now, suppose

\[
S: (s, t) \rightarrow S(s, t), \quad 0 \leq s \leq 1, \quad 0 \leq t \leq 1
\]

is a surface in $X$ such that: for fixed $s$, $t \rightarrow S(s, t)$ is vertical, i.e. lies in a fibre, and for fixed $t$, $s \rightarrow S(s, t)$ is horizontal.

Suppose $S^*(w_i) = A_i(s, t)dt + B_i(s, t)ds$. The geometric interpretation of these functions is that $A_i(\theta_0(S(s, t)))$ resp. $B_i(\theta_0(S(s, t)))$ is the tangent vector to the curve $t \rightarrow S(s, t)$ resp. $s \rightarrow S(s, t)$. Then

\[(4.4)\]
\[A_u = 0, \quad B_a = 0.\]

From 5.1 we derive

\[(4.5)\]
\[\frac{\partial}{\partial s} A_i - \frac{\partial}{\partial t} B_i = f_{jki}(S(s, t))B_jA_k.\]

To prove that $B$ inherits a Riemannian metric from the given metric on $X$, it suffices to show that, for every such surface $S$,

\[(4.6)\]
\[\frac{\partial}{\partial t} B_u B_v = 0\]

for then it would follow that the lengths of all horizontal liftings of a given tangent vector to $B$ have the same length.

Using (4.4) and (4.5), we have

\[\frac{\partial}{\partial t} B_u B_v = 2B_u c_{au} B_v A_a.\]

For each $a$, the matrix $c_{au}$ is the matrix of Ad $g_a$ acting on $M$: by our assumption it is skew-symmetric, hence

\[B_u c_{au} B_v = 0.\]

This proves (4.6).

We want to now show that the fibres of $\phi$ are totally geodesic. Let
then $\sigma : [0, 1] \rightarrow X$ be a curve in the fibre, i.e. $\sigma'(t)$ is vertical, and $v : t \rightarrow v(t) \subseteq X_{\sigma(t)}$ a vertical vector field along $\sigma$, i.e. $w_u(v(t)) = 0$. From (4.3) we have

$$w_u(\Delta v) = -h_{uab}(\sigma(t))w_b(\sigma''(t))w_u(v(t)).$$

From 4.2

$$h_{uab} = \frac{1}{2} (c_{bau} - c_{aub} + c_{uab}).$$

That each of these three terms is zero follows from the relations $[K, K] \subset K$, $[K, M] \subset M$.

BIBLIOGRAPHY