ON BOREL'S METHOD OF SUMMABILITY

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1. Introduction. Let be given the series
\[ \sum a_k, \text{ with partial sums } s_k = a_0 + \cdots + a_k. \]
Throughout the paper, \( k \) runs through the integers 0, 1, \( \ldots \) and \( \sum \) stands for \( \sum_{k=0}^{\infty} \). The radius of convergence of the power series \( \sum a_k s_k \) shall be denoted by \( \rho \). If \( \rho > 0 \), we put \( f(z) = \sum a_k s_k \) for \( |z| < \rho \).

The following theorem about Borel's summability method \( B \) [6, p. 182; 10, p. 134] is well known.

**Theorem A.** If \( \sum a_k \) is summable \( B \) and \( 0 < \rho < 1 \), then \( f(z) \) can be continued analytically onto the disc \( |z-1/2| < 1/2 \).

\[ \sum a_k \] is called regularly (singularly) summable \( B \) if it is summable \( B \) and \( \rho > 0 \) (\( \rho = 0 \)). Using functional analytic concepts, we show in §3 that for each prescribed \( \rho \) \( (0 < \rho < 1) \) there exists a series \( \sum a_k \) which is regularly summable \( B \) and for which \( f(z) \) cannot be continued analytically beyond the boundary of the union of the discs \( |z| < \rho \) and \( |z-1/2| < 1/2 \). Analogously we deal with the case of singular summability.

In §4 it is pointed out that the method \( B \) is not equivalent with any row-finite matrix method. This is a consequence of the fact that the \( FK \)-space of all series \( \sum a_k \) which are summable \( B \) is not a \( BK \)-space. About \( FK \)- and \( BK \)-spaces cf. [10, p. 29].

Gaier [4] investigated the discrete variant \( B_1 \) of \( B \). (For typographical reasons, we use \( B_1 \) instead of Gaier's \( B_f \).) The definition of \( B_1 \) is repeated in §2. A main result of Gaier is

**Theorem B.** If \( \sum a_k \) is summable \( B_1 \) and if there is a constant \( K \), \( 0 < K < (\pi^2 + 1)^{1/2} \), such that \( a_k = O(K^k) \) for \( k \to \infty \), then \( \sum a_k \) is summable \( B \).

Continuing Gaier's investigation of the method \( B_1 \), we see in §4 that also \( B_1 \), which is a row-infinite matrix method, is not equivalent with any row-finite matrix method.

In §5 we put the question whether there is, for \( B_1 \), a theorem in the direction of Theorem A. We get the result that if \( \sum a_k \) is summable \( B_1 \) and \( 0 < \rho < 1 \) then \( f(z) \) is regular in a certain disc containing the point \( z = \rho \) in its interior. The proof uses Gaier's main tool, a theorem

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of Cartwright on radial limits of entire functions.

In §6 we are concerned with gap theorems. First, it is shown how the regularity theorem of §5 yields a new proof of, and further insight into another theorem of Gaier of the $B_1 \to B$ type. Second, an assertion of Erdős is proved by a method formerly used by the authors in the case of Taylor's method of summability.

2. Preliminaries. We state the definitions of the methods $B$ and $B_1$, together with other known facts needed afterwards. (For references see §1.) We keep the notations of §1 and introduce some new ones.

The Borel method $B$ connects with $\sum a_k$ the transform

$$b(x) = e^{-x} \sum s_k x^k / k! = a_0 + \int_0^x e^{-t}a(t) dt, \quad a(t) = \sum a_k + t^k / k!.$$ 

$\sum a_k$ is called summable $B$ to the value $s$ if $b(x)$ exists for $x \geq 0$, i.e. if $a(z)$ is an entire function, and if $b(x) \to s$ as $x \to \infty$. We identify $\sum a_k$ with the sequence $\mathcal{A} = \{a_k\}$. The convergence domain (Wirkfeld) of the method $B$, consisting of all $\mathcal{A}$ for which $\sum a_k$ is summable $B$, shall be denoted by $\mathcal{B}$. The distinction between regular and singular summability splits $\mathcal{B}$ into two disjoint nonempty subsets: $\mathcal{B} = \mathcal{B}^r + \mathcal{B}^s$.

If $\mathcal{A} \in \mathcal{B}$ the Laplace integral $\int_0^\infty e^{-zt}a(t) dt$ is convergent for $z = 1$, and therefore for $\Re z > 1$ ($\Re$ means: real part). Let the disc $|z - 1/2| < 1/2$ be denoted by $D$, its boundary by $C$.

If $\mathcal{A} \in \mathcal{B}^r$, $f(z)$ is defined and regular in the union of the discs $|z| < \rho$ and $D$, moreover

$$(2.1) \quad f(1/z) = a_0 + \int_0^\infty e^{-zt}a(t) dt \quad (\Re z > 1).$$

If $\mathcal{A} \in \mathcal{B}^s$, we use (2.1) to define $f(1/z)$ for $\Re z > 1$. With each $\mathcal{A} \in \mathcal{B}$ there is now associated a function $f(z)$ which is regular at least in $D$. $\mathcal{A} \in \mathcal{B}$ implies $\sum a_k z^k = f(z)$ for $0 < z < 1$.

Gaier's modification of the method $B$, yielding the method $B_1$, consists in replacing $\lim_{x \to \infty} b(x)$ by $\lim_{n \to \infty} b(n)$, where $n = 0, 1, \cdots$. In other words, $B_1$ is the sequence-to-sequence matrix defined by the matrix $(a_{nk})$,

$$(2.2) \quad a_{nk} = e^{-n^k / k!} \quad (n, k = 0, 1, \cdots, a_{00} = 1).$$

It is trivial that $\mathcal{B} \subset \mathcal{B}_1$, where $\mathcal{B}_1$ is the convergence domain of $B_1$, and indeed $\mathcal{B} \neq \mathcal{B}_1$. If $\mathcal{A} \in \mathcal{B}_1$, then $b(x)$ and $a(t)$ have the same meaning as before. It is clear how we define regular and singular summability $B_1$ and how we understand the decomposition $\mathcal{B}_1 = \mathcal{B}_1^r + \mathcal{B}_1^s$. 

If \( \mathcal{A} \in \mathcal{B}_1^R \), then \( \rho > 0 \) and \( f(z) = \sum a_k z^k \) is defined and regular for \( |z| < \rho \). We do not define generally a function \( f(z) \) for a \( \mathcal{A} \in \mathcal{B}_S^R \).

The Borel method \( B^* \) connects with \( \sum a_k \) the transform

\[ b^*(x) = \int_0^x e^{-a^*(t)} dt, \quad \text{where} \quad a^*(t) = \sum a_k t^k / k! . \]

\( \sum a_k \) is called summable \( B^* \) to the value \( s \) if \( a^*(z) \) is an entire function and if \( b^*(x) \rightarrow s \) for \( x \rightarrow \infty \). The corresponding discrete variant \( B_1^* \) of \( B^* \) has also been considered by Gaier. (Gaier uses the notation \( B' \) instead of \( B^* \).) The subsequent treatment of \( B \) and \( B_1 \) is analogously admitted by \( B^* \) and \( B_1^* \). In most cases it is sufficient to observe that summability \( B^* \) (\( B_1^* \)) of the series \( b_0 + b_1 + \cdots \) is equivalent with summability \( B \) (\( B_1 \)) of the series \( 0 + b_0 + b_1 + \cdots \). For instance, Theorems 130 and 132 in [6, pp. 185–186] (or IV and V in [10, pp. 135–136]) imply that Theorem A is true for \( B^* \) instead of \( B \), hence Theorem A holds. We shall not mention \( B^* \) and \( B_1^* \) anymore.

3. Noncontinuability. Two interesting examples of elements \( \mathcal{A} \in \mathcal{B}_S^R \) were given by Hardy [6, p. 189]. In both cases the domain of regularity of \( f(z) \) is larger than \( D = \{ \left| z - 1/2 \right| < 1/2 \} \). We show now that there are many \( \mathcal{A} \in \mathcal{B} \) for which \( f(z) \) has \( C = \{ \left| z - 1/2 \right| = 1/2 \} \) as its natural boundary.

**Theorem 1.** There is an element \( \mathcal{A} \in \mathcal{B}_S^R \) such that \( f(z) \) cannot be continued analytically beyond \( C \).

We only sketch the proof which follows standard lines. \( \mathcal{B} \) is a \( F \)-space whose topology is given by the semi-norms

\[ \rho(\mathcal{A}) = \sup_{x > 0} |b(x)|, \quad \rho_j(\mathcal{A}) = \sum_j f^k |a_k| / k! \quad (j = 1, 2, \ldots) \]

(see e.g. Włodarski [9]; cf. [11] and §4). Since the mappings \( \mathcal{A} \rightarrow a_k \) are continuous linear functionals, \( \mathcal{B} \) is a \( FK \)-space. Given any point \( w \) of the exterior of \( D \) there are elements \( \mathcal{A}_0 \in \mathcal{B} \) such that \( f_0(z) \) is singular at \( z = w \). The usual condensation procedure (cf. e.g. [12, p. 421, 10.5 and 10.6]) yields the \( \mathcal{A} \) in question.

The proof even shows that the set of those elements \( \mathcal{A} \in \mathcal{B} \) for which \( f(z) \) can be continued analytically beyond \( C \) is of the first category in \( \mathcal{B} \). It follows that \( \mathcal{B}_R^S \) is of the first, and \( \mathcal{B}_S^R \) of the second category in \( \mathcal{B} \).

The same method of proof yields

**Theorem 2.** Given \( \rho_0 \) (\( 0 < \rho_0 < 1 \)) there is an element \( \mathcal{A} \in \mathcal{B}_R^S \) with \( \rho = \rho_0 \) and such that \( f(z) \) cannot be continued analytically beyond the boundary of the union of the discs \( |z| < \rho_0 \) and \( D \).
4. Nonequivalence. A FK-space is called a BK-space if its topology can be given by a single norm. It is of interest to know whether or not $\mathcal{B}$ is a BK-space. The following theorem gives a negative answer involving an inequivalence theorem.

**Theorem 3.** The FK-space $\mathcal{B}$ is not a BK-space. The method $B$ is not equivalent to any row-finite matrix method.

A corresponding theorem is true for $\mathcal{B}_1$ and $B_1$. Since the proofs run for both cases analogously, we restrict ourselves to the method $B_1$.

First of all, we introduce a $F$-topology in the convergence domain $\mathcal{B}_1$. The subsequent lemma is easily obtained. (See e.g. [12, p. 414], where a similar theorem is given for functions regular in the unit circle.)

**Lemma 1.** The set of elements $\mathcal{A}$, for which $b(z)$ is an entire function, is a FK-space with any one of the following three systems of semi-norms:

\[
q_j(\mathcal{A}) = \sup_{|z| = j} |b(z)| \quad (j = 1, 2, \ldots),
\]

\[
\bar{q}_j(\mathcal{A}) = \sup_k \left| \sum_{m=0}^{k} j^m s_m / m! \right| \quad (j = 1, 2, \ldots),
\]

\[
\hat{q}_j(\mathcal{A}) = \sup_k j^k s_k / k! \quad (j = 1, 2, \ldots).
\]

Each of these three systems introduces the same topology.

We put

\[
q(\mathcal{A}) = \sup_{n=1,2,\ldots} |b(n)|.
\]

Then it follows from Lemma 1 (cf. [12, p. 294, 2.1 and 2.2]) that $\mathcal{B}_1$ is a FK-space with anyone of the following three (topologically equivalent) systems of semi-norms: $[q, q_j], [q, \bar{q}_j], [q, \hat{q}_j]$ ($j = 1, 2, \ldots$).

**Theorem 4.** The FK-space $\mathcal{B}_1$ is not a BK-space. The method $B_1$ is not equivalent to any row-finite matrix method.

Intending an indirect proof of the first part of Theorem 4, we assume that $\mathcal{B}_1$ is a BK-space. Then there exist [10, p. 30, VI] positive numbers $\Omega_j$ and a natural number $m$ such that

\[
q_j(\mathcal{A}) \leq \Omega_j (q(\mathcal{A}) + q_1(\mathcal{A}) + \cdots + q_m(\mathcal{A})) \quad (\mathcal{A} \in \mathcal{B}_1; j = 1, 2, \ldots).
\]

This can easily be disproved by functions of the form $b(z) = e^{-az}$ 

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where $\alpha > 0$ is large. Using the first part of Theorem 4, the second part can be proved by known arguments. The reader may check e.g. [8, p. 37], where the proof of an analogous statement is carried out in detail.

5. Regularity. Since for a general entire function $b(z)$ the condition $b(n) \to 0$ ($n = 0, 1, \cdots ; n \to \infty$) is a rather weak one we cannot expect for $B_1$ a full analogue of Theorem A. There is, however, the following

**Theorem 5.** If $\sum a_k$ is regularly summable $B_1$ and $0 < \rho < 1$ then $f(z)$ can be continued analytically onto the disc $|z - c| < c$, where

$$c = \begin{cases} 
2^{-1}(\sigma^2 - \pi^2)^{-1/2} & \text{if } \sigma = \rho^{-1} \geq (\pi^2 + 1)^{1/2}, \\
2^{-1} & \text{if } \sigma = \rho^{-1} < (\pi^2 + 1)^{1/2}.
\end{cases}$$

If $\sigma < (\pi^2 + 1)^{1/2}$, then the conclusion of the theorem follows from Theorems A and B. The following proof treats both cases simultaneously.

Let be $\mathfrak{A} \subseteq \mathfrak{B}_{\sigma}$, $0 < \rho < 1$, and $a_0 = 0$. The entire function $a(z)$ is of order 1 and type $\sigma$, and the relation

$$\int_0^\infty e^{-w^2}a(t)dt = \sum a_{k+1}w^{-k-1} = f(1/w)$$

holds at least for $\Re w > \sigma$ [1, p. 73]. We shall show that the Laplace transform in (5.1) exists in a larger domain of the $w$-plane from which the asserted regularity property will follow.

Integration by parts yields

$$\int_0^\infty e^{-w^2}a(t)dt = e^{-(w-1)x}b(x) + (w - 1) \int_0^\infty e^{-(w-1)t}b(t)dt.$$ 

We put

$$g(x) = e^{-(w-1)x}b(x),$$

where $\omega$ is a fixed real number $> 1$. Since $g(n) \to 0$ ($n = 0, 1, \cdots ; n \to \infty$) and using Cartwright's theorem ([1, p. 180]; for further references see [4, p. 874]), we deduce that

$$g(x) \to 0 \quad (x > 0 \text{ real}, x \to \infty)$$

if there is a number $\alpha$ ($0 < \alpha \leq \pi/2$) for which the indicator function $h(\theta)$ of $g(z)$ satisfies the condition $h(\pm \alpha) < \pi \sin \alpha$. We find easily

$$h(\theta) \leq \sigma - \omega \cos \theta \quad (0 \leq \theta < 2\pi).$$

It follows that (5.3) is true if for a suitable $\alpha$ we have
\[ \omega > \phi(\alpha), \quad \text{where} \quad \phi(\alpha) = (\sigma - \pi \sin \alpha) / \cos \alpha. \]

If \( \sigma \geq (\pi^2 + 1)^{1/2} \), \( \phi(\alpha) \) has the smallest value \( (\sigma^2 - \pi^2)^{1/2} \) (taken for \( \sin \alpha = \pi / \sigma \)); if \( \sigma < (\pi^2 + 1)^{1/2} \), there are values \( \phi(\alpha) \) which are <1. Therefore (5.3) is certainly true if \( \omega > (\sigma^2 - \pi^2)^{1/2} \) in the first case, and always in the second case.

It follows now from (5.2) that the Laplace integral in (5.1) exists in the half-plane

\[ \Re w > \begin{cases} (\sigma^2 - \pi^2)^{1/2} & \text{if } \sigma \geq (\pi^2 + 1)^{1/2}, \\ 1 & \text{if } \sigma < (\pi^2 + 1)^{1/2}. \end{cases} \]

Herewith Theorem 5 is proved.

If \( \sigma \geq (\pi^2 + 1)^{1/2} \), the number \( c \) of Theorem 5 cannot be replaced by a bigger one, as can be seen by the following example. (Examples of this kind were used by Gaier [4] for similar purposes.) Let \( \tau \geq 1 \). We define \( \sum a_k \) by the equation

\[ b(x) = e^{(\tau-1)x} \sin \pi x \quad (x > 0). \]

Then \( \sum a_k \) is summable \( B_1 \) (to the value 0) and we have

\[ s_k = (2i)^{-1}((\tau + i\pi)^k - (\tau - i\pi)^k), \quad \sigma = \rho^{-1} = (\tau^2 + \pi^2)^{1/2}, \]

\[ f(z) = (1 - z) \sum s_k z^k = (1 - z)(2i)^{-1}[(1 - (\tau + i\pi)z)^{-1} - (1 - (\tau - i\pi)z)^{-1}], \quad (|z| < \rho). \]

We prescribe now for \( \sigma \) a value \( \geq (\pi^2 + 1)^{1/2} \), which means that we have to take \( \tau = (\sigma^2 - \pi^2)^{1/2} \). \( f(z) \) has the singularities \( z = (\tau \pm i\pi)^{-1} \), these points being the intersection points of \( |z| = \rho \) and \( |z - c| = c \). (Observe that \( |z| = \sigma \) and \( \Re z = (\sigma^2 - \pi^2)^{1/2} \) intersect in \( z = \tau \pm i\pi \).) Therefore \( f(z) \) is regular in \( |z - c| < c \), and not in \( |z - d| < d \) for \( d > c \).

Using known geometric properties of the Borel summability polygon, we deduce from Theorem 5 immediately

**Theorem 6.** If \( \sum a_k \) is regularly summable \( B_1 \) and \( 0 < \rho \leq (\pi^2 + 1)^{-1/2} \), then \( \sum a_k z^k \) is regularly summable \( B \) for \( 0 \leq z < (\sigma^2 - \pi^2)^{-1/2} (\sigma = \rho^{-1}) \).

The remaining case \( (\pi^2 + 1)^{-1/2} < \rho \leq 1 \) of this theorem is settled by Theorem B; then \( \sum a_k z^k \) is regularly summable \( B \) for \( 0 \leq z \leq 1 \). Theorem 6, together with Theorem A, yields back the case \( \sigma \geq (\pi^2 + 1)^{1/2} \) of Theorem 5.

6. **Gaps.** \( \sum a_k \) is said to be a Fabry gap series if \( a_k = 0 \) for \( k \neq k_m \), where \( \{ k_m \} \) is a sequence of integers, \( 0 \leq k_0 < k_1 < \cdots \), and \( k_m / m \to \infty \) for \( m \to \infty \).
The regularity theorem of §5 makes possible a new approach to the following theorem of Gaier [5, p. 496].

**Theorem 7.** If \( \sum a_k \) is a Fabry gap series and is regularly summable \( B_1 \), then it is regularly summable \( B \).

It follows from Theorem 5 and Fabry’s gap theorem that a Fabry gap series \( \sum a_k \) cannot be regularly summable \( B_1 \) unless \( \rho \geq 1 \). Herewith Theorem 7 is reduced to Theorem B.

Finally we are concerned with a statement of Erdős [3, p. 267]: There exists, for \( B_1 \), no pure Tauberian gap theorem. A theorem of this kind, indeed for Taylor’s method \( T_{\alpha} \), was given also by the authors [7, p. 223; 8, p. 49]. Erdős gave no proof for his assertion. We show now that our method of proof in the \( T_{\alpha} \) case, depending on a result of Eidelheit and Pólya, can be used to prove Erdős’ theorem which runs as follows.

**Theorem 8.** Given any sequence \( \{ k_m \} \) of integers, \( 0 \leq k_0 < k_1 < \cdots \), then there exists a divergent series \( \sum a_k \) which is summable \( B_1 \) and for which \( a_k = 0 \) for \( k \neq k_m \) \( (m = 0, 1, \cdots) \).

**Proof.** First of all, we observe that the sequence-to-sequence matrix method \( B_1 \), defined by the matrix (2.2), is equivalently given in series-to-sequence form by the matrix \( (b_{nk}) \),

\[
b_{nk} = e^{-n} \sum_{j=k}^{\infty} n^j/j! \quad (n, k = 0, 1, \cdots; b_{00} = 1).
\]

Particularly, if \( \sum a_k \) is summable \( B_1 \) then \( b(n) = \sum b_{nk} a_k \) \( (n = 0, 1, \cdots) \). Since, for \( n = 0, 1, \cdots \) and \( k = 1, 2, \cdots \),

\[
b_{nk} = (\Gamma(k))^{-1} \int_{0}^{n} e^{-t} t^{k-1} dt
\]

and

\[
0 < b_{nk}/b_{n+1,k} \leq e^{n+1} \left( \int_{0}^{n} t^{k-1} dt \right) \left( \int_{0}^{n+1} t^{k-1} dt \right)^{-1} = e^{n+1} n^k/(n+1)^k,
\]

the matrix \( (c_{nm}) \),

\[
c_{nm} = b_{nk_m} \quad (n, m = 0, 1, \cdots),
\]

has the property that, for each fixed \( n = 1, 2, \cdots \), \( c_{nm}/c_{n+1,m} \to 0 \) for \( m \to \infty \). Using results of Eidelheit, Pólya, and Banach [2, p. 32; 10, p. 33, III and p. 32, II] we conclude that the system of equations \( \sum_{m=1}^{\infty} c_{nm} x_m = 0 \) \( (n = 1, 2, \cdots) \) has an infinite number of solutions.
\{x_1, x_2, \ldots \}$. Let the sequence \{\tilde{x}_1, \tilde{x}_2, \ldots \} be such a solution, not with all \( \tilde{x}_m = 0 \). Putting \( \tilde{x}_0 = 0 \) and observing \( c_{01} = c_{02} = \cdots = 0 \), we have \( \sum_{m=0}^{\infty} c_{nm} \tilde{x}_m = 0 \) \((n = 0, 1, \ldots)\). Let \( a_k = \tilde{x}_m \) for \( k = k_m \), and \( a_k = 0 \) for \( k \neq k_m \) \((m = 0, 1, \ldots)\). The series \( \sum a_k \) which is now defined satisfies the gap condition under consideration and is summable \( B_1 \) since

\[
(6.1) \quad b(n) = \sum_{m=0}^{\infty} b_{nk_m} a_{k_m} = \sum_{m=0}^{\infty} c_{nm} \tilde{x}_m = 0 \quad (n = 0, 1, \ldots).
\]

All we have still to do is, to show that \( \sum a_k \) is divergent. We assume that \( \sum a_k \) is convergent, i.e. that \( \{s_k\} \) is convergent. Then we have for the entire function \( b(z) \) the estimate \( b(z) = O(e^{2\pi |z|}) \) for \( |z| \to \infty \). By (6.1) and the uniqueness theorem of Carlson it follows \([1, \text{p. 153, 9.2.1, p. 75, 5.4.1}]\) that \( b(z) \) is identically zero, implying \( a_k = 0 \) for all \( k \). Since not all \( \tilde{x}_m \) are zero we get a contradiction which proves the theorem.

**References**


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