# ON BOREL'S METHOD OF SUMMABILITY

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**1. Introduction.** Let be given the series

$$\sum a_k, \text{ with partial sums } s_k = a_0 + \cdots + a_k.$$

Throughout the paper, $k$ runs through the integers $0, 1, \cdots$ and $\sum$ stands for $\sum_{k=0}^{\infty}$. The radius of convergence of the power series $\sum a_k z^k$ shall be denoted by $\rho$. If $\rho > 0$, we put $f(z) = \sum a_k z^k$ for $|z| < \rho$. The following theorem about Borel's summability method $B$ [**6**, p. 182; **10**, p. 134] is well known.

THEOREM A. *If $\sum a_k$ is summable $B$ and $0 < \rho < 1$, then $f(z)$ can be continued analytically onto the disc $|z - 1/2| < 1/2$.*

$\sum a_k$ is called regularly (singularly) summable $B$ if it is summable $B$ and $\rho > 0$ ($\rho = 0$). Using functional analytic concepts, we show in §3 that for each prescribed $\rho$ $(0 < \rho < 1)$ there exists a series $\sum a_k$ which is regularly summable $B$ and for which $f(z)$ cannot be continued analytically beyond the boundary of the union of the discs $|z| < \rho$ and $|z - 1/2| < 1/2$. Analogously we deal with the case of singular summability.

In §4 it is pointed out that the method $B$ is not equivalent with any row-finite matrix method. This is a consequence of the fact that the $FK$-space of all series $\sum a_k$ which are summable $B$ is not a $BK$-space. About $FK$- and $BK$-spaces cf. [**10**, p. 29].

Gaier [**4**] investigated the discrete variant $B_1$ of $B$. (For typographical reasons, we use $B_1$ instead of Gaier's $B_I$.) The definition of $B_1$ is repeated in §2. A main result of Gaier is

THEOREM B. *If $\sum a_k$ is summable $B_1$ and if there is a constant $K$, $0 < K < (\pi^2 + 1)^{1/2}$, such that $a_k = O(K^k)$ for $k \to \infty$, then $\sum a_k$ is summable $B$.*

Continuing Gaier's investigation of the method $B_1$, we see in §4 that also $B_1$, which is a row-infinite matrix method, is not equivalent with any row-finite matrix method.

In §5 we put the question whether there is, for $B_1$, a theorem in the direction of Theorem A. We get the result that if $\sum a_k$ is summable $B_1$ and $0 < \rho < 1$ then $f(z)$ is regular in a certain disc containing the point $z = \rho$ in its interior. The proof uses Gaier's main tool, a theorem

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of Cartwright on radial limits of entire functions.

In §6 we are concerned with gap theorems. First, it is shown how the regularity theorem of §5 yields a new proof of, and further insight into another theorem of Gaier of the $B_1 \to B$ type. Second, an assertion of Erdös is proved by a method formerly used by the authors in the case of Taylor's method of summability.

2. **Preliminaries.** We state the definitions of the methods $B$ and $B_1$, together with other known facts needed afterwards. (For references see §1.) We keep the notations of §1 and introduce some new ones.

The Borel method $B$ connects with $\sum a_k$ the transform

$$b(x) = e^{-x} \sum s_k x^k/k! = a_0 + \int_0^x e^{-t} a(t)dt, \qquad a(t) = \sum a_{k+1} t^k/k!.$$

$\sum a_k$ is called summable $B$ to the value $s$ if $b(x)$ exists for $x \geqq 0$, i.e. if $a(z)$ is an entire function, and if $b(x) \to s$ as $x \to \infty$. We identify $\sum a_k$ with the sequence $\mathfrak{A} = \{a_k\}$. The convergence domain (Wirkfeld) of the method $B$, consisting of all $\mathfrak{A}$ for which $\sum a_k$ is summable $B$, shall be denoted by $\mathfrak{B}$. The distinction between regular and singular summability splits $\mathfrak{B}$ into two disjoint nonempty subsets: $\mathfrak{B} = \mathfrak{B}^R + \mathfrak{B}^S$.

If $\mathfrak{A} \in \mathfrak{B}$ the Laplace integral $\int_0^\infty e^{-zt} a(t)dt$ is convergent for $z=1$, and therefore for $\Re z > 1$ ($\Re$ means: real part). Let the disc $|z - 1/2| < 1/2$ be denoted by $D$, its boundary by $C$. If $\mathfrak{A} \in \mathfrak{B}^R$, $f(z)$ is defined and regular in the union of the discs $|z| < \rho$ and $D$, moreover

$$f(1/z) = a_0 + \int_0^\infty e^{-zt} a(t)dt \qquad (\Re z > 1). \tag{2.1}$$

If $\mathfrak{A} \in \mathfrak{B}^S$, we use (2.1) to define $f(1/z)$ for $\Re z > 1$. With each $\mathfrak{A} \in \mathfrak{B}$ there is now associated a function $f(z)$ which is regular at least in $D$. $\mathfrak{A} \in \mathfrak{B}$ implies $B$-$\sum a_k z^k = f(z)$ for $0 < z < 1$.

Gaier's modification of the method $B$, yielding the method $B_1$, consists in replacing $\lim_{x\to\infty} b(x)$ by $\lim_{n\to\infty} b(n)$, where $n = 0, 1, \cdots$. In other words, $B_1$ is the sequence-to-sequence matrix method defined by the matrix $(a_{nk})$,

$$a_{nk} = e^{-n} n^k/k! \qquad (n, k = 0, 1, \cdots, a_{00} = 1). \tag{2.2}$$

It is trivial that $\mathfrak{B} \subset \mathfrak{B}_1$, where $\mathfrak{B}_1$ is the convergence domain of $B_1$, and indeed $\mathfrak{B} \neq \mathfrak{B}_1$. If $\mathfrak{A} \in \mathfrak{B}_1$, then $b(x)$ and $a(t)$ have the same meaning as before. It is clear how we define regular and singular summability $B_1$ and how we understand the decomposition $\mathfrak{B}_1 = \mathfrak{B}_1^R + \mathfrak{B}_1^S$.

If $\mathfrak{A}\in\mathfrak{B}_1^R$, then $\rho>0$ and $f(z)=\sum a_k z^k$ is defined and regular for $|z|<\rho$. We do not define generally a function $f(z)$ for a $\mathfrak{A}\in\mathfrak{B}_1^S$.

The Borel method $B^*$ connects with $\sum a_k$ the transform

$$b^*(x) = \int_0^x e^{-t}a^*(t)dt, \quad \text{where} \quad a^*(t) = \sum a_k t^k/k!.$$

$\sum a_k$ is called summable $B^*$ to the value $s$ if $a^*(z)$ is an entire function and if $b^*(x)\to s$ for $x\to\infty$. The corresponding discrete variant $B_1^*$ of $B^*$ has also been considered by Gaier. (Gaier uses the notation $B'$ instead of $B^*$.) The subsequent treatment of $B$ and $B_1$ is analogously admitted by $B^*$ and $B_1^*$. In most cases it is sufficient to observe that summability $B^*$ ($B_1^*$) of the series $b_0+b_1+\cdots$ is equivalent with summability $B$ ($B_1$) of the series $0+b_0+b_1+\cdots$. For instance, Theorems 130 and 132 in [**6**, pp. 185–186] (or IV and V in [**10**, pp. 135–136]) imply that Theorem A is true for $B^*$ instead of $B$, hence Theorem A holds. We shall not mention $B^*$ and $B_1^*$ anymore.

3. **Noncontinuability.** Two interesting examples of elements $\mathfrak{A}\in\mathfrak{B}^S$ were given by Hardy [**6**, p. 189]. In both cases the domain of regularity of $f(z)$ is larger than $D=\{|z-1/2|<1/2\}$. We show now that there are many $\mathfrak{A}\in\mathfrak{B}$ for which $f(z)$ has $C=\{|z-1/2|=1/2\}$ as its natural boundary.

THEOREM 1. *There is an element $\mathfrak{A}\in\mathfrak{B}^S$ such that $f(z)$ cannot be continued analytically beyond $C$.*

We only sketch the proof which follows standard lines. $\mathfrak{B}$ is a $F$-space whose topology is given by the semi-norms

$$p(\mathfrak{A}) = \sup_{x>0} |b(x)|, \qquad p_j(\mathfrak{A}) = \sum j^k |a_k|/k! \qquad (j=1,2,\cdots)$$

(see e.g. Włodarski [**9**]; cf. [**11**] and §4). Since the mappings $\mathfrak{A}\to a_k$ are continuous linear functionals, $\mathfrak{B}$ is a $FK$-space. Given any point $w$ of the exterior of $D$ there are elements $\mathfrak{A}_0\in\mathfrak{B}$ such that $f_0(z)$ is singular at $z=w$. The usual condensation procedure (cf. e.g. [**12**, p. 421, 10.5 and 10.6]) yields the $\mathfrak{A}$ in question.

The proof even shows that the set of those elements $\mathfrak{A}\in\mathfrak{B}$ for which $f(z)$ can be continued analytically beyond $C$ is of the first category in $\mathfrak{B}$. It follows that $\mathfrak{B}^R$ is of the first, and $\mathfrak{B}^S$ of the second category in $\mathfrak{B}$.

The same method of proof yields

THEOREM 2. *Given $\rho_0$ $(0<\rho_0<1)$ there is an element $\mathfrak{A}\in\mathfrak{B}^R$ with $\rho=\rho_0$ and such that $f(z)$ cannot be continued analytically beyond the boundary of the union of the discs $|z|<\rho_0$ and $D$.*

**4. Nonequivalence.** A $FK$-space is called a $BK$-space if its topology can be given by a single norm. It is of interest to know whether or not $\mathfrak{B}$ is a $BK$-space. The following theorem gives a negative answer involving an inequivalence theorem.

THEOREM 3. *The FK-space $\mathfrak{B}$ is not a BK-space. The method B is not equivalent to any row-finite matrix method.*

A corresponding theorem is true for $\mathfrak{B}_1$ and $B_1$. Since the proofs run for both cases analogously, we restrict ourselves to the method $B_1$.

First of all, we introduce a $F$-topology in the convergence domain $\mathfrak{B}_1$. The subsequent lemma is easily obtained. (See e.g. [**12**, p. 414], where a similar theorem is given for functions regular in the unit circle.)

LEMMA 1. *The set of elements $\mathfrak{A}$, for which $b(z)$ is an entire function, is a FK-space with any one of the following three systems of semi-norms:*

$$q_j(\mathfrak{A}) = \sup_{|z|=j} |b(z)| \qquad (j = 1, 2, \cdots),$$

$$\bar{q}_j(\mathfrak{A}) = \sup_k \left| \sum_{m=0}^{k} j^m s_m/m! \right| \qquad (j = 1, 2, \cdots),$$

$$\hat{q}_j(\mathfrak{A}) = \sup_k j^k |s_k|/k! \qquad (j = 1, 2, \cdots).$$

*Each of these three systems introduces the same topology.*

We put

$$q(\mathfrak{A}) = \sup_{n=1,2,\ldots} |b(n)|.$$

Then it follows from Lemma 1 (cf. [**12**, p. 294, 2.1 and 2.2]) that $\mathfrak{B}_1$ is a $FK$-space with anyone of the following three (topologically equivalent) systems of semi-norms: $[q, q_j]$, $[q, \bar{q}_j]$, $[q, \hat{q}_j]$ $(j=1, 2, \cdots)$.

THEOREM 4. *The FK-space $\mathfrak{B}_1$ is not a BK-space. The method $B_1$ is not equivalent to any row-finite matrix method.*

Intending an indirect proof of the first part of Theorem 4, we assume that $\mathfrak{B}_1$ is a $BK$-space. Then there exist [**10**, p. 30, VI] positive numbers $\Omega_j$ and a natural number $m$ such that

$$q_j(\mathfrak{A}) \leqq \Omega_j(q(\mathfrak{A}) + q_1(\mathfrak{A}) + \cdots + q_m(\mathfrak{A})) \quad (\mathfrak{A} \in \mathfrak{B}_1; j = 1, 2, \cdots).$$

This can easily be disproved by functions of the form $b(z) = e^{-\alpha z}$

where $\alpha>0$ is large. Using the first part of Theorem 4, the second part can be proved by known arguments. The reader may check e.g. [**8**, p. 37], where the proof of an analogous statement is carried out in detail.

5. **Regularity.** Since for a general entire function $b(z)$ the condition $b(n)\to 0$ $(n=0, 1, \cdots; n\to\infty)$ is a rather weak one we cannot expect for $B_1$ a full analogue of Theorem A. There is, however, the following

THEOREM 5. *If $\sum a_k$ is regularly summable $B_1$ and $0<\rho<1$ then $f(z)$ can be continued analytically onto the disc $|z-c|<c$, where*

$$c = \begin{cases} 2^{-1}(\sigma^2 - \pi^2)^{-1/2} & \text{if } \sigma = \rho^{-1} \geqq (\pi^2+1)^{1/2}, \\ 2^{-1} & \text{if } \sigma = \rho^{-1} < (\pi^2+1)^{1/2}. \end{cases}$$

If $\sigma<(\pi^2+1)^{1/2}$, then the conclusion of the theorem follows from Theorems A and B. The following proof treats both cases simultaneously.

Let be $\mathfrak{A}\in\mathfrak{B}_1^R$, $0<\rho<1$, and $a_0=0$. The entire function $a(z)$ is of order 1 and type $\sigma$, and the relation

$$\int_0^\infty e^{-wt}a(t)dt = \sum a_{k+1}w^{-k-1} = f(1/w) \tag{5.1}$$

holds at least for $\Re w>\sigma$ [**1**, p. 73]. We shall show that the Laplace transform in (5.1) exists in a larger domain of the $w$-plane from which the asserted regularity property will follow.

Integration by parts yields

$$\int_0^x e^{-wt}a(t)dt = e^{-(w-1)x}b(x) + (w-1)\int_0^x e^{-(w-1)t}b(t)dt. \tag{5.2}$$

We put

$$g(x) = e^{-(\omega-1)x}b(x),$$

where $\omega$ is a fixed real number $>1$. Since $g(n)\to 0$ $(n=0, 1, \cdots; n\to\infty)$ and using Cartwright's theorem ([**1**, p. 180]; for further references see [**4**, p. 874]), we deduce that

$$g(x)\to 0 \qquad (x>0 \text{ real}, x\to\infty) \tag{5.3}$$

if there is a number $\alpha$ $(0<\alpha\leqq\pi/2)$ for which the indicator function $h(\theta)$ of $g(z)$ satisfies the condition $h(\pm\alpha)<\pi\sin\alpha$. We find easily

$$h(\theta) \leqq \sigma - \omega\cos\theta \qquad (0\leqq\theta<2\pi).$$

It follows that (5.3) is true if for a suitable $\alpha$ we have

$$\omega > \phi(\alpha), \quad \text{where} \quad \phi(\alpha) = (\sigma - \pi \sin \alpha)/\cos \alpha.$$

If $\sigma \geqq (\pi^2+1)^{1/2}$, $\phi(\alpha)$ has the smallest value $(\sigma^2-\pi^2)^{1/2}$ (taken for $\sin \alpha = \pi/\sigma$); if $\sigma < (\pi^2+1)^{1/2}$, there are values $\phi(\alpha)$ which are $<1$. Therefore (5.3) is certainly true if $\omega > (\sigma^2-\pi^2)^{1/2}$ in the first case, and always in the second case.

It follows now from (5.2) that the Laplace integral in (5.1) exists in the half-plane

$$\Re w > \begin{cases} (\sigma^2 - \pi^2)^{1/2} & \text{if } \sigma \geqq (\pi^2 + 1)^{1/2}, \\ 1 & \text{if } \sigma < (\pi^2 + 1)^{1/2}. \end{cases}$$

Herewith Theorem 5 is proved.

If $\sigma \geqq (\pi^2+1)^{1/2}$, the number $c$ of Theorem 5 cannot be replaced by a bigger one, as can be seen by the following example. (Examples of this kind were used by Gaier [4] for similar purposes.) Let $\tau \geqq 1$. We define $\sum a_k$ by the equation

$$b(x) = e^{(\tau-1)x} \sin \pi x \qquad (x > 0).$$

Then $\sum a_k$ is summable $B_1$ (to the value 0) and we have

$$\begin{aligned} s_k &= (2i)^{-1}((\tau + i\pi)^k - (\tau - i\pi)^k), \qquad \sigma = \rho^{-1} = (\tau^2 + \pi^2)^{1/2}, \\ f(z) &= (1 - z) \sum s_k z^k \\ &= (1 - z)(2i)^{-1}[(1 - (\tau + i\pi)z)^{-1} - (1 - (\tau - i\pi)z)^{-1}], \\ &\qquad\qquad (|z| < \rho). \end{aligned}$$

We prescribe now for $\sigma$ a value $\geqq (\pi^2+1)^{1/2}$, which means that we have to take $\tau = (\sigma^2-\pi^2)^{1/2}$. $f(z)$ has the singularities $z = (\tau \pm i\pi)^{-1}$, these points being the intersection points of $|z| = \rho$ and $|z-c| = c$. (Observe that $|z| = \sigma$ and $\Re z = (\sigma^2-\pi^2)^{1/2}$ intersect in $z = \tau \pm i\pi$.) Therefore $f(z)$ is regular in $|z-c| < c$, and not in $|z-d| < d$ for $d > c$.

Using known geometric properties of the Borel summability polygon, we deduce from Theorem 5 immediately

THEOREM 6. *If $\sum a_k$ is regularly summable $B_1$ and $0 < \rho \leqq (\pi^2+1)^{-1/2}$, then $\sum a_k z^k$ is regularly summable $B$ for $0 \leqq z < (\sigma^2-\pi^2)^{-1/2}$ $(\sigma = \rho^{-1})$.*

The remaining case $(\pi^2+1)^{-1/2} < \rho \leqq 1$ of this theorem is settled by Theorem B; then $\sum a_k z^k$ is regularly summable $B$ for $0 \leqq z \leqq 1$. Theorem 6, together with Theorem A, yields back the case $\sigma \geqq (\pi^2+1)^{1/2}$ of Theorem 5.

**6. Gaps.** $\sum a_k$ is said to be a Fabry gap series if $a_k = 0$ for $k \neq k_m$, where $\{k_m\}$ is a sequence of integers, $0 \leqq k_0 < k_1 < \cdots$, and $k_m/m \to \infty$ for $m \to \infty$.

The regularity theorem of §5 makes possible a new approach to the following theorem of Gaier [**5**, p. 496].

THEOREM 7. *If $\sum a_k$ is a Fabry gap series and is regularly summable $B_1$, then it is regularly summable $B$.*

It follows from Theorem 5 and Fabry's gap theorem that a Fabry gap series $\sum a_k$ cannot be regularly summable $B_1$ unless $\rho \geqq 1$. Herewith Theorem 7 is reduced to Theorem B.

Finally we are concerned with a statement of Erdös [**3**, p. 267]: There exists, for $B_1$, no pure Tauberian gap theorem. A theorem of this kind, indeed for Taylor's method $T_\alpha$, was given also by the authors [**7**, p. 223; **8**, p. 49]. Erdös gave no proof for his assertion. We show now that our method of proof in the $T_\alpha$ case, depending on a result of Eidelheit and Pólya, can be used to prove Erdös' theorem which runs as follows.

THEOREM 8. *Given any sequence $\{k_m\}$ of integers, $0 \leqq k_0 < k_1 < \cdots$, then there exists a divergent series $\sum a_k$ which is summable $B_1$ and for which $a_k = 0$ for $k \neq k_m$ $(m = 0, 1, \cdots)$.*

PROOF. First of all, we observe that the sequence-to-sequence matrix method $B_1$, defined by the matrix (2.2), is equivalently given in series-to-sequence form by the matrix $(b_{nk})$,

$$b_{nk} = e^{-n} \sum_{j=k}^{\infty} n^j/j! \qquad (n, k = 0, 1, \cdots; b_{00} = 1).$$

Particularly, if $\sum a_k$ is summable $B_1$ then $b(n) = \sum b_{nk} a_k$ $(n = 0, 1, \cdots)$. Since, for $n = 0, 1, \cdots$ and $k = 1, 2, \cdots$,

$$b_{nk} = (\Gamma(k))^{-1} \int_0^n e^{-t} t^{k-1} dt$$

and

$$0 < b_{nk}/b_{n+1;k} \leqq e^{n+1} \left( \int_0^n t^{k-1} dt \right) \left( \int_0^{n+1} t^{k-1} dt \right)^{-1} = e^{n+1} n^k/(n+1)^k,$$

the matrix $(c_{nm})$,

$$c_{nm} = b_{nk_m} \qquad (n, m = 0, 1, \cdots),$$

has the property that, for each fixed $n = 1, 2, \cdots$, $c_{nm}/c_{n+1,m} \to 0$ for $m \to \infty$. Using results of Eidelheit, Pólya, and Banach [**2**, p. 32; **10**, p. 33, III and p. 32, II] we conclude that the system of equations $\sum_{m=1}^{\infty} c_{nm} x_m = 0$ $(n = 1, 2, \cdots)$ has an infinite number of solutions

$\{x_1, x_2, \cdots\}$. Let the sequence $\{\bar{x}_1, \bar{x}_2, \cdots\}$ be such a solution, not with all $\bar{x}_m = 0$. Putting $\bar{x}_0 = 0$ and observing $c_{01} = c_{02} = \cdots = 0$, we have $\sum_{m=0}^{\infty} c_{nm}\bar{x}_m = 0$ $(n = 0, 1, \cdots)$. Let $a_k = \bar{x}_m$ for $k = k_m$, and $a_k = 0$ for $k \neq k_m$ $(m = 0, 1, \cdots)$. The series $\sum a_k$ which is now defined satisfies the gap condition under consideration and is summable $B_1$ since

$$b(n) = \sum_{m=0}^{\infty} b_{nk_m} a_{k_m} = \sum_{m=0}^{\infty} c_{nm}\bar{x}_m = 0 \qquad (n = 0, 1, \cdots). \tag{6.1}$$

All we have still to do is, to show that $\sum a_k$ is divergent. We assume that $\sum a_k$ is convergent, i.e. that $\{s_k\}$ is convergent. Then we have for the entire function $b(z)$ the estimate $b(z) = O(e^{2|z|})$ for $|z| \to \infty$. By (6.1) and the uniqueness theorem of Carlson it follows [1, p. 153, 9.2.1, p. 75, 5.4.1] that $b(z)$ is identically zero, implying $a_k = 0$ for all $k$. Since not all $\bar{x}_m$ are zero we get a contradiction which proves the theorem.

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