SOME SPECIAL TRANSFORMATION GROUPS

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Suppose H is an abelian, compact, connected, metric group. Then an (onto) isometry of H, relative to some translation invariant metric, has an extremely simple form. Indeed

THEOREM 1. *If ϕ is a self-homeomorphism of H preserving some invariant metric, then $\phi(h)=h_1\sigma(h)$, $h\in H$, where σ is an automorphism of H. Furthermore if H is finite dimensional σ is periodic.*

Since one can trivially construct an invariant metric preserved by a given periodic automorphism, one has the corollary that an automorphism of such a finite dimensional H is periodic if and only if it preserves some invariant metric. However, our restriction that H be metric is entirely artificial; our basic result is

THEOREM 2. *Let H be a compact connected abelian group, and let G be an equicontinuous group of self-homeomorphisms of H containing all translations. Then $g\in G$ is of the form $g(h)=h_g\sigma_g(h)$, where σ_g is an automorphism of H. If H is also finite dimensional there is an integer k for which $\sigma_g^k(h)=h$ for all g and h.*

The proof uses some elementary vector-valued integration, the fact that on compact groups one has approximate identities consisting of trigonometric polynomials (by the Peter-Weyl and Stone-Weierstrass theorems), as well as one conceivably new (but trivial) fact (§1, Lemma). Thus our arguments will be rather conventional. The results arose from an attempt to apply some work of K. de Leeuw and the author [1, Theorem 1] and undoubtedly bear traces of conversations with de Leeuw, to whom the author would like to express his thanks.

1. **A remark on connectedness.** As is well known a compact abelian group is connected if and only its character group $H^\wedge$ has no torsion (i.e., no elements of finite order); and it is a standard result that a torsion free abelian group can be (totally) ordered so as to become an ordered abelian group.[1] Thus when H is compact, connected, and

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[1] That is, the set of elements greater than the identity is closed under the group operation.

abelian we can assume $H^\wedge$ an ordered abelian group, obtaining the following simple consequence.

LEMMA. *A unimodular trigonometric polynomial on a compact connected abelian group H is a monomial, i.e., if* $\left|\sum_{i=1}^n a_i\hat{h}_i\right| = 1$, $\hat{h}_i \in H^\wedge$, $\hat{h}_i \neq \hat{h}_j$ for $i \neq j$, *then exactly one* $a_i \neq 0$.

PROOF. We can suppose all $a_i \neq 0$, $n > 1$, and that the $\hat{h}_i$ are indexed as ordered: $\hat{h}_1 < \hat{h}_2 < \cdots < \hat{h}_n$. Since $\left|\sum_{i=1}^n a_i\hat{h}_i\right| = 1$, $(\sum_{i=1}^n a_i\hat{h}_i)(\sum_{i=1}^n \bar{a}_i\hat{h}_i^{-1}) = 1 = a_1\bar{a}_n\hat{h}_1\hat{h}_n^{-1} +$ terms involving larger characters; since $\hat{h}_1\hat{h}_n^{-1}$ is not the identity character we conclude that $a_1\bar{a}_n = 0$, our contradiction.[2]

2. **Proof of Theorem** 2. By Ascoli's theorem our group G has compact closure in the space of all continuous maps of H into itself, taken in the compact-open topology. Moreover it is easily seen that its closure is again a group of self-homeomorphisms, and a compact topological group in the compact-open topology. Using this topology $(g, h) \to g(h)$ is continuous from $G \times H \to H$, and we may as well assume G is a compact (effective) transformation group acting on H [2].

The elements of G induce a group $\mathcal{G}$ of maps $f \to f \circ g$ taking $C(H)$ onto itself, and trivially $g \to f \circ g$ is a continuous map of G into $C(H)$ for each $f \in C(H)$. Consequently for any $v \in C(G)$ we can form the vector-valued (Haar) integral

$$\int v(g) f \circ g dg. \tag{1}$$

Let $\{v\}$ be an approximate identity on G consisting of trigonometric polynomials (i.e. each v is a finite linear combination of entries in finite dimensional matricial representations of G). Each v lies in a finite dimensional translation invariant subspace of $C(G)$, and if $v_1, v_2, \cdots, v_n$ span this subspace, the corresponding integrals $\int v_i(g) f \circ g dg$, $i = 1, 2, \cdots, n$, span a finite dimensional subspace of $C(H)$ containing (1) which is $\mathcal{G}$-invariant. Indeed

[2] The lemma remains valid if compactness is omitted; for the almost periodic compactification of H is then a compact connected abelian group on which our trigonometric polynomial is unimodular. Indeed replacing "trigonometric polynomial" by "finite linear combination of multiplicative characters" we can remove "abelian" as well, using an analogous compactification (the dual of the discrete group of "multiplicative characters"). Finally note that this property characterizes connected groups among the compact abelian groups. For if H is disconnected $H^\wedge$ must have some element $\hat{h}$ of finite order which maps H onto the nth roots of unity for some $n > 1$, whence $\hat{h} + \hat{h}^2 + \cdots + \hat{h}^n = 0$ and $|\hat{h} + \hat{h}^2 + \cdots + \hat{h}^{n-1}| = 1$.

$$\left(\int v_i(g)f\circ g dg\right)\circ g_1 = \int v_i(g)f\circ(gg_1)dg = \int v_i(gg_1^{-1})f\circ g dg$$

$$= \int \sum_{j=1}^{n} c_j v_j(g)f\circ g dg = \sum_{j=1}^{n} c_j \int v_j(g)f\circ g dg.$$

But for any $\epsilon>0$ we can choose v so that f lies at a distance $<\epsilon$ from (1) in $C(H)$; consequently $C(H)$ is the closed span of finite dimensional $\mathcal{G}$-invariant subspaces $\{V\}$.

Now since G contains all translations each V is a translation invariant subspace of $C(H)$ and thus is spanned by characters $\hat{h}_1, \hat{h}_2, \cdots, \hat{h}_n$ of H; since the subspaces V span $C(H)$ each $\hat{h}\in H^\wedge$ must of course appear in some such basis. And since $\hat{h}_i\circ g=\sum_{j=1}^{n} c_j\hat{h}_j$ is a unimodular trigonometric polynomial we see from our lemma that $\hat{h}_i\circ g=\lambda\hat{h}_j$, where λ is a unimodular complex number. Thus for each $\hat{h}\in H^\wedge$ and fixed $g\in G$ we have $\hat{h}\circ g=\lambda_{\hat{h}}\tau_g(\hat{h})$, where $\tau_g(\hat{h})\in H^\wedge$; clearly $\hat{h}\to\lambda_{\hat{h}}$ is a character of $H^\wedge$ and we may write $\hat{h}\circ g(h) =(g(h), \hat{h})=(h_g, \hat{h})(h, \tau_g(\hat{h}))$, or $(h_g^{-1}g(h), \hat{h})=(h, \tau_g(\hat{h}))$. Hence $h\to(h_g^{-1}g(h), \hat{h})$ is multiplicative for each $\hat{h}$, so that $\sigma_g: h\to h_g^{-1}g(h)$ is a homomorphism, therefore an automorphism, as asserted. Indeed $\tau_g: H^\wedge\to H^\wedge$ is clearly the automorphism dual to σ_g.

For the final assertion of Theorem 2 note that each τ_g acts simply as a permutation on the basis characters of each V. Thus if V is n-dimensional, for any g, $\tau_g^{n!}$ leaves each character in V's basis fixed. If H is finite dimensional, so that [3] we have $\hat{h}_1, \hat{h}_2, \cdots, \hat{h}_m$ in $H^\wedge$ for which $\hat{h}\in H^\wedge$ implies $\hat{h}^n=\hat{h}_1^{n_1}\hat{h}_2^{n_2}\cdots\hat{h}_m^{n_m}$, we can find an integer k (the product of appropriate factorials) for which $\tau_g^k\hat{h}_i=\hat{h}_i$, all i, g, and thus assert that $\tau_g^k\hat{h}=\hat{h}$, all $\hat{h}$, g. For the set of $\hat{h}$ satisfying this equality clearly is closed under the group operation and (when they exist) the taking of roots: if $\tau_g^k(\hat{h})=\hat{h}$ and $(\hat{h}')^j=\hat{h}$ then $(\tau_g^k\hat{h}')^j=\tau_g^k\hat{h} =\hat{h}=(\hat{h}')^j$ so that $\tau_g^k\hat{h}'=\hat{h}'$ (since $H^\wedge$ has no elements of finite order). Thus, dually, σ_g^k is the identity map for all $g\in G$, completing our proof.

Theorem 1 is of course an immediate consequence since ϕ and the group of translations then generate an equicontinuous group of self-homeomorphisms.

3. Groups of automorphisms. If we restrict our self-homeomorphisms to automorphisms we can obtain some apparently new results when H is finite dimensional.

THEOREM 3. *Let H be a finite dimensional compact connected abelian group, G an equicontinuous group of automorphisms of H. Then for some integer k, every element of G is of period k.*

For the proof note that we may as well assume G is compact in the compact-open topology, as in §2. Thus if ρ is any translation invariant metric on H, then setting

$$d(h_1, h_2) = \sup_g \rho(g(h_1), g(h_2))$$

yields an equivalent metric d which is clearly G-invariant, and is also translation invariant:

$$d(hh_1, hh_2) = \sup_g \rho(g(hh_1), g(hh_2)) = \sup_g \rho(g(h)g(h_1), g(h)g(h_2))$$
$$= \sup_g \rho(g(h_1), g(h_2)) = d(h_1, h_2).$$

Consequently G and the translations generate an equicontinuous group to which Theorem 2 applies.[3]

COROLLARY. *If H is a finite dimensional, compact, connected, normal, abelian subgroup of a maximally almost periodic group G, there is an integer k for which*[4] $g^k h = hg^k$, $g \in G$, $h \in H$.

If G were taken compact we should only need to apply Theorem 3 to the group $\{\sigma_g : g \in G\}$, where $\sigma_g : h \to g^{-1}hg$. But if G is maximally almost periodic we have H mapped in a (1-1) continuous fashion into the almost periodic compactification G^* of G, and thus (homeomorphically and) isomorphically; since H must remain normal in G^*, clearly, $g^k h = hg^k$ for all $g \in G^*$.

REFERENCES

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2. D. Montgomery and L. Zippin, *Topological transformation groups*, New York, 1955.

3. A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Paris, 1940.

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[3] It should perhaps be noted that while our first use of connectedness in the proof of Theorem 2 evaporates in the special case of automorphisms, there is a second which remains.

[4] As an example where H is not central let G be the group of all isometries of the circle group T^1, H the normal subgroup formed by the translations, where $k=2$. More generally we can replace T^1 by any compact connected metric abelian group; trivially the translations form a normal subgroup of the group of isometries, while $x \to x^{-1}$ is a nontrivial isometric automorphism.