HOMOGENEOUS GAMES. II

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Introduction. This paper describes strong simple homogeneous n-player games for several values of n of the form $2^k(2^l-1)$, $l>k$; specifically, for the (Mersenne) primes 2^l-1 and for the first two composite values, 15, 63 (for any $k<l$). The problem of the existence of such a game remains open for $n=20, 24, 40, \cdots$.

Let us call the games *fair* games for short. Heuristically, a fair game of n players is a rule for deciding disputed binary questions without giving any one player an advantage—for example, majority rule, if n is odd. Arrow's theorem on the nonexistence of a social welfare function [**2**] asserts in effect that for questions which are more than binary, no fair complete rule is possible.

Precisely, a fair game on a set N of players is a family of subsets of N, called winning sets, such that (a) every set containing a winning set is winning, (b) the complement of a winning set is not winning, (c) the complement of a nonwinning set is winning, and (d) the group of all permutations of N which take winning sets to winning sets is transitive.

The problem of constructing a fair game reduces at once [**1**, Lemma 1] to the problem of constructing its group: a transitive group of permutations, every element of which has at least one odd cycle. We recall from [**1**] that the class of all n for which a fair n-player game exists is closed under multiplication and contains the odd n and the $n\equiv 2 \pmod 4$, except 2. Impossibility is known only for n a power of 2 (except 1) and $n=12$.

1. The construction utilizes the finite projective space $P=PG(2, l-1)$ over the two-element field. Observe that P has a collineation permuting its 2^l-1 points cyclically [**3**, pp. 384–385].

LEMMA. *If 2^l-1 is prime, 15, or 63, then $PG(2, l-1)$ admits a transitive collineation group Z of odd order such that for any z in Z and any $l-1$ hyperplanes H_i in P, there is $p\in P$ such that the number of points common to the orbit of p under powers of z and H_i, for each i, is odd.*

PROOF. Let Z be a cyclic collineation group as in [**3**]. Specifically, for $l=4$ and $l=6$, we take x^4+x+1 and x^6+x+1 as the irreducible polynomials in Singer's construction.

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If z is a generator of Z, the orbit of any p is all of P and the intersection with every hyperplane is odd. If z is the identity, choose p common to all H_i. For primes 2^l-1, there is no other case. For the case $l=4$, computation shows that the exceptional orbits are (a) lines (3 points) and (b) skew pentagons V such that every plane containing two points of V contains exactly three points of V. Each kind of orbit has odd intersection with every plane. The same thing happens for $l=6$; all exceptional orbits are unions of odd numbers of (a) lines or (b) planes. This establishes the lemma.

I do not know whether the lemma remains valid for $PG(2, 7)$ or for other spaces of composite order.

2. For any l satisfying the conditions of the lemma, for any $k<l$, we construct first a group H of functions on $P=PG(2, l-1)$ which may be described as the direct sum of k copies of the group of complements of hyperplanes. Precisely, let S_0 denote the empty set, and $S_1, \cdots, S_m$ $(m=2^l-1)$ the complements of hyperplanes in P. The sets S_j form a group under symmetric difference, since the symmetric difference of the complements of two hyperplanes intersecting in an $(l-3)$-subspace T is the complement of the third hyperplane through T. Let K be the direct sum of k copies of Z_2, with generators $a_1, \cdots, a_k$. In the group K^P of all functions on P to K, let f_{ij} $(i=1, \cdots, k; j=0, \cdots, m)$ denote the function which takes the value a_i on S_j and 0 on its complement. (All f_{i0} vanish.) Let H be the subgroup generated by these functions. Then every element of H has the form $\sum_{i=1}^{i=k} f_{ij(i)}$; for these functions include the generators f_{ij} and are closed under addition. (The group is commutative, and $f_{ir}+f_{is}=f_{it}$ for suitable t.)

Next let Q be an index set of 2^k elements and select a transitive action of K on Q. (For example, let Q be a product of k two-element sets and let a_i operate by changing every ith coordinate.) On the product set $P\times Q$, of $2^k m$ elements, we define an action of H by $h(p, q)=(p, h(p)(q))$. Let Z be a group acting on P as in the lemma, and let Z act on $P\times Q$ by $z(p, q)=(z(p), q)$. Let G be the least group of permutations of $P\times Q$ containing H and Z.

Since the group of functions H is invariant under collineations of P, Z is contained in the normalizer of H and every element of G can be written (uniquely) in the form hz. Explicitly, $hz(p, q)=(z(p), h(z(p))(q))$, and $(hz)^s(p, q)=(z^s(p), [\sum_{r=1}^{r=s} h(z^r(p))](q))$. Now the order of z is an odd number, and every cycle of z is odd. As for h, it is a sum of k or fewer functions f_{ij}; by the lemma, there is p in P such that the number of points common to the orbit of p under powers

of z and each S_j is even. Let s be the number of points of the orbit and q any index in Q. For $r<s$, $(hz)^r(p, q)$ differs from (p, q) in the first coordinate; but $(hz)^s(p, q)=(p, q)$. Thus every element of G has an odd cycle. As we noted above, this implies [1] the existence of a fair game of $2^k(2^l-1)$ players.

References

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ON INDUCED TOPOLOGIES IN QUASI-REFLEXIVE BANACH SPACES[1]

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1. **Introduction.** Let π denote the canonical isomorphism of a Banach space X into its second conjugate space X^{**}. An example is given by James [4] of a space X for which X is separable, X is not reflexive, X is isomorphic to X^{**}, and $X^{**}/\pi(X)$ is one-dimensional. Civin and Yood undertook a more complete investigation of Banach spaces X such that $X^{**}/\pi(X)$ is (finite) n-dimensional and called such spaces quasi-reflexive Banach spaces of order n. If Q is a subset of X^*, let $\sigma(X, Q)$ denote the least fine topology for X such that all $x^*\in Q$ are continuous. In [1] Civin and Yood establish the following result.

Theorem A. *The following statements are equivalent:*

(1) *X is quasi-reflexive of order n.*

(2) *There is an equivalent norm for X such that $X^*=Q\oplus R$ where Q is a total closed linear manifold such that the unit ball of X is compact in $\sigma(X, Q)$ and R is an n-dimensional linear manifold.*

It is the purpose of this paper to study properties of the topologies $\sigma(X, Q)$, where $X^*=Q\oplus R$, Q is a total closed linear manifold, and

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