HOMOGENEOUS GAMES. II

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Introduction. This paper describes strong simple homogeneous \( n \)-player games for several values of \( n \) of the form \( 2^k(2^l-1) \), \( l > k \); specifically, for the (Mersenne) primes \( 2^l-1 \) and for the first two composite values, 15, 63 (for any \( k < l \)). The problem of the existence of such a game remains open for \( n = 20, 24, 40, \ldots \).

Let us call the games fair games for short. Heuristically, a fair game of \( n \) players is a rule for deciding disputed binary questions without giving any one player an advantage—for example, majority rule, if \( n \) is odd. Arrow's theorem on the nonexistence of a social welfare function [2] asserts in effect that for questions which are more than binary, no fair complete rule is possible.

Precisely, a fair game on a set \( N \) of players is a family of subsets of \( N \), called winning sets, such that (a) every set containing a winning set is winning, (b) the complement of a winning set is not winning, (c) the complement of a nonwinning set is winning, and (d) the group of all permutations of \( N \) which take winning sets to winning sets is transitive.

The problem of constructing a fair game reduces at once [1, Lemma 1] to the problem of constructing its group: a transitive group of permutations, every element of which has at least one odd cycle. We recall from [1] that the class of all \( n \) for which a fair \( n \)-player game exists is closed under multiplication and contains the odd \( n \) and the \( n = 2 \) (mod 4), except 2. Impossibility is known only for \( n \) a power of 2 (except 1) and \( n = 12 \).

1. The construction utilizes the finite projective space \( P = PG(2, l-1) \) over the two-element field. Observe that \( P \) has a collineation permuting its \( 2^l-1 \) points cyclically [3, pp. 384–385].

Lemma. If \( 2^l-1 \) is prime, 15, or 63, then \( PG(2, l-1) \) admits a transitive collineation group \( Z \) of odd order such that for any \( z \) in \( Z \) and any \( l-1 \) hyperplanes \( H_i \) in \( P \), there is \( p \in P \) such that the number of points common to the orbit of \( p \) under powers of \( z \) and \( H_i \), for each \( i \), is odd.

Proof. Let \( Z \) be a cyclic collineation group as in [3]. Specifically, for \( l=4 \) and \( l=6 \), we take \( x^4+x+1 \) and \( x^6+x+1 \) as the irreducible polynomials in Singer's construction.

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If \( z \) is a generator of \( Z \), the orbit of any \( p \) is all of \( P \) and the intersection with every hyperplane is odd. If \( z \) is the identity, choose \( p \) common to all \( H_i \). For primes \( 2^l-1 \), there is no other case. For the case \( l=4 \), computation shows that the exceptional orbits are (a) lines (3 points) and (b) skew pentagons \( V \) such that every plane containing two points of \( V \) contains exactly three points of \( V \). Each kind of orbit has odd intersection with every plane. The same thing happens for \( l=6 \); all exceptional orbits are unions of odd numbers of (a) lines or (b) planes. This establishes the lemma.

I do not know whether the lemma remains valid for \( PG(2, 7) \) or for other spaces of composite order.

2. For any \( l \) satisfying the conditions of the lemma, for any \( k<l \), we construct first a group \( H \) of functions on \( P = PG(2, l-1) \) which may be described as the direct sum of \( k \) copies of the group of complements of hyperplanes. Precisely, let \( S_0 \) denote the empty set, and \( S_1, \ldots, S_m \) \( (m=2^l-1) \) the complements of hyperplanes in \( P \). The sets \( S_i \) form a group under symmetric difference, since the symmetric difference of the complements of two hyperplanes intersecting in an \((l-3)\)-subspace \( T \) is the complement of the third hyperplane through \( T \). Let \( K \) be the direct sum of \( k \) copies of \( Z_2 \), with generators \( a_1, \ldots, a_k \). In the group \( K^P \) of all functions on \( P \) to \( K \), let \( f_{ij} \) \( (i=1, \ldots, k; j=0, \ldots, m) \) denote the function which takes the value \( a_i \) on \( S_j \) and 0 on its complement. (All \( f_{i0} \) vanish.) Let \( H \) be the subgroup generated by these functions. Then every element of \( H \) has the form \( \sum f_{ij}(a_i) \); for these functions include the generators \( f_{ij} \) and are closed under addition. (The group is commutative, and \( f_{ir} + f_{is} = f_{ir} \) for suitable \( t \).)

Next let \( Q \) be an index set of \( 2^k \) elements and select a transitive action of \( K \) on \( Q \). (For example, let \( Q \) be a product of \( k \) two-element sets and let \( a_i \) operate by changing every \( i \)th coordinate.) On the product set \( P \times Q \), of \( 2^km \) elements, we define an action of \( H \) by \( h(p, q) = (p, h(p)(q)) \). Let \( Z \) be a group acting on \( P \) as in the lemma, and let \( Z \) act on \( P \times Q \) by \( z(p, q) = (z(p), q) \). Let \( G \) be the least group of permutations of \( P \times Q \) containing \( H \) and \( Z \).

Since the group of functions \( H \) is invariant under collineations of \( P \), \( Z \) is contained in the normalizer of \( H \) and every element of \( G \) can be written (uniquely) in the form \( hz \). Explicitly, \( hz(p, q) = (z(p), h(z(p))(q)) \), and \( (hz)^*(p, q) = (z^*(p), \sum h(z^*(p)))(q) \). Now the order of \( z \) is an odd number, and every cycle of \( z \) is odd. As for \( h \), it is a sum of \( k \) or fewer functions \( f_{ij} \); by the lemma, there is \( p \) in \( P \) such that the number of points common to the orbit of \( p \) under powers
of $z$ and each $S_j$ is even. Let $s$ be the number of points of the orbit and $q$ any index in $Q$. For $r < s$, $(hz)^r(\phi, q)$ differs from $(\phi, q)$ in the first coordinate; but $(hz)^s(\phi, q) = (\phi, q)$. Thus every element of $G$ has an odd cycle. As we noted above, this implies [1] the existence of a fair game of $2^s(2^r - 1)$ players.

References


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ON INDUCED TOPOLOGIES IN QUASI-REFLEXIVE BANACH SPACES

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1. Introduction. Let $\pi$ denote the canonical isomorphism of a Banach space $X$ into its second conjugate space $X^{**}$. An example is given by James [4] of a space $X$ for which $X$ is separable, $X$ is not reflexive, $X$ is isomorphic to $X^{**}$, and $X^{**}/\pi(X)$ is one-dimensional. Civin and Yood undertook a more complete investigation of Banach spaces $X$ such that $X^{**}/\pi(X)$ is (finite) $n$-dimensional and called such spaces quasi-reflexive Banach spaces of order $n$. If $Q$ is a subset of $X^*$, let $\sigma(X, Q)$ denote the least fine topology for $X$ such that all $x^* \in Q$ are continuous. In [1] Civin and Yood establish the following result.

Theorem A. The following statements are equivalent:

1. $X$ is quasi-reflexive of order $n$.
2. There is an equivalent norm for $X$ such that $X^* = Q \oplus R$ where $Q$ is a total closed linear manifold such that the unit ball of $X$ is compact in $\sigma(X, Q)$ and $R$ is an $n$-dimensional linear manifold.

It is the purpose of this paper to study properties of the topologies $\sigma(X, Q)$, where $X^* = Q \oplus R$, $Q$ is a total closed linear manifold, and

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