REMARKS ON CAUCHY'S INTEGRAL FORMULA IN MATRIC SPACES

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1. Introduction. Recently several proofs of Cauchy's integral formula have been given for matric spaces [2; 4; 5; 7]. However a short direct proof is available by using the argument that Morita gives to prove the Poisson formula (see §2). In §3 the formula is also proved by means of a minimal problem, similar to those introduced by Bergman [1]. Since the present paper is closely related to Morita's [7], we use his notation wherever possible.

The matric spaces under consideration are the four main types of irreducible bounded symmetric domains given by E. Cartan [3]. Let \( z \) be a matrix of complex numbers, \( z' \) its transpose, \( z^* \) its conjugate transpose and \( I(r) \) the identity matrix of order \( r \). Then the first three types are defined by

\[
D = E[z \mid I(n) - z^*z > 0],
\]

where

I. \( \mathbb{A}_{mn} \): \( z \) is a matrix of type \((m, n)(m \geq n)\).

II. \( \mathbb{S}_n \): \( z \) is a symmetric matrix of order \( n \).

III. \( \mathcal{Q}_n \): \( z \) is a skew symmetric matrix of order \( n \).

The fourth type is

IV. \( \mathbb{M}_n \): the set of all matrices \( z \) of type \((n, 1)\) (that is, \( n \)-dimensional vectors) such that

\[
|z'z| < 1, \quad 1 - 2z^*z + |z'z|^2 > 0.
\]

It is known that each of the domains possesses a distinguished boundary \( B \) [1], which is defined by

\[
z^*z = I(n)
\]

for \( \mathbb{A}_{mn} \), \( \mathbb{S}_n \) and for \( \mathcal{Q}_n \) if \( n \) is even, or the eigenvalues of \( z^*z \) are all 1 except one which is zero if \( n \) is odd. For \( \mathbb{M}_n \), \( B \) is given by

\[
z^*z = 1, \quad |z'| = 1.
\]

2. Cauchy's integral formula. We define a kernel function (the Cauchy kernel) by

\[
K(z, \xi) = V^{-1} \det^{-p} (z - \xi),
\]

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299
where \( p = n \) for domains \( \mathbb{H}_n \), \((n+1)/2\) for \( \mathbb{S}_n \), \((n-1)/2\) for \( \mathbb{L}_n \) if \( n \) is even, and \( V \) is the Euclidean volume of the domain \( B \). For domains \( \mathcal{M}_n \)

\[
K(z, \xi) = V^{-1}[(z - \xi)'(z - \xi)]^{-n/2}.
\]

Then

**Theorem 1.** Let \( f(z) \) be regular in \( D \) and continuous on \( \overline{D} \) (the closure of \( D \)), where \( D \) is one of the domains \( \mathbb{H}_n, \mathbb{S}_n, \mathbb{L}_n \) or \( \mathcal{M}_n \). Then

\[
f(\xi) = \int_B K(\xi, z)f(z)\hat{z}, \quad \xi \in D,
\]

where

\[
\hat{z} = c_{n1} \prod_{j=1}^{n} dz_{jk}, \quad \text{for } \mathbb{H}_n
\]

\[
= c_{n2} \prod_{j=1; k \neq j}^{n} dz_{jk}, \quad \text{for } \mathbb{S}_n
\]

\[
= c_{n3} \prod_{j=1}^{n-1} dz_{jk}, \quad \text{for } \mathbb{L}_n \text{ (} n \text{ even)}
\]

\[
= c_{n4} \prod_{j=1}^{n} dz_{j}, \quad \text{for } \mathcal{M}_n,
\]

and the constants \( c_{nj} \) are such that \( V^{-1}\int_B K^{-1}(z, 0)\hat{z} = 1 \) in each case. (We note that \( K^{-1}(z, 0)\hat{z} \) is the Euclidean volume element for the set \( B \) [7].)

**Proof.** We shall restrict ourselves to the case \( D = \mathbb{H}_n \) but the other cases may be treated similarly. See [7, §16] for the details in the case \( \mathcal{M}_n \).

It is known that the set of analytic mappings taking \( D \) onto itself and \( \xi \) into 0 and the inverse transformations are given by [8]

\[
w = a(z - \xi)(d - d*z)^{-1},
\]

\[
z = \sigma(w) = (a + wd*\xi)^{-1}(wd + a\xi)
= (a*w + \xi d*)(d* + \xi*a*w)^{-1},
\]

subject to the conditions

\[
a(I - \xi a*)a* = I, \quad d(I - \xi d*)d* = I, \quad \xi*a*a = d*d\xi*,
\]

\[
a*a - \xi d*d\xi* = I, \quad d*d - \xi*a*a\xi* = I,
\]

\((I = I^{(n)})\). Also these transformations leave the set \( B \) invariant.
Suppose first that \( f(z) \) is analytic on \( \bar{D} \) and consider the expression

\[
F(\xi, \tau, z^*, \bar{z}) = f(z) \det^{-n} (I - z^*\xi)d\nu_z,
\]
where \( d\nu_z \) is the Euclidean volume element of the set \( B \):

\[
d\nu_z = \det^{-n} d\bar{z}.
\]

Under the transformation (9)

\[
\frac{\partial(z)}{\partial(w)} = \det^{-n} (a + w\xi^*) \det^{-n} (d^* + \xi^*a^*w).
\]

Also since \( w^*w = I \),

\[
d\nu_z = \det^{-n} (d + w^*\xi^*) \det^{-n} (d^* + \xi^*a^*w)d\nu_w,
\]
and

\[
I - z^*\xi = (d + w^*a^*)^{-1}d(I - \xi^*\xi).
\]

Thus

\[
\int_B F(\xi, \tau, z^*, \bar{z}) = \det^{-n} d \det^{-n} (I - \xi^*\xi) \int_B f_0(w)d\nu_w.
\]

where

\[
f_0(w) = \det^{-n} (d^* + \xi^*a^*w)f(\sigma(w))
\]
is regular on \( \bar{D} \).

By a theorem due to H. Cartan a regular function on \( \bar{D} \) can be expanded on \( \bar{D} \) into a uniformly convergent series of homogeneous polynomials, \( \sum_{n=0}^\infty a_nP_n(w) \), where \( P_0(w) \) is a constant so that \( a_0P_0 = f_0(0) \) and \( P_n \) is of degree \( >0 \) for \( n>0 \). Also since \( B \) is circular

\[
\int_B P_n(w)d\nu_w = 0 \quad \text{for } n > 0.
\]

Thus

\[
\int_B f_0(w)d\nu_w = Vf_0(0)
\]
and by (9a) and (10)
\[ (1/V) \int_B F(\xi, z, z^*, \hat{z}) = f(\xi). \]

Since on \( z^*z = I \),
\[ \text{det}^{-n} (I - z^*\xi) dv_z = \text{det}^{-n} (z - \xi) \hat{z}, \]
(7) follows for functions regular on \( \overline{D} \).

In case \( f(z) \) is regular on \( D \) and continuous on \( \overline{D} \), following Morita, we have
\[ f(t\xi) = \int_B K(\xi, z)f(tz) dv_z \]
for any real number \( t \) such that \( 0 \leq t < 1 \). Letting \( t \to 1^- \) we see that (7) holds for such a function \( f(z) \). Thus Theorem 1 is proved.

3. Minimal problem. Let \( D \) be an arbitrary bounded domain in the space of \( n \) complex variables with a distinguished boundary \( \partial D \). Let \( \xi \) be an arbitrary fixed point of \( D \) and consider the subclass \( S \) of regular functions \( f \) on \( \overline{D} \) such that \( f(\xi) = 1 \). Suppose there exists a function \( M(\xi, z) \) of \( S \) which minimizes the integral
\[ \int_B |f(z)|^2 dv_z, \quad \in S. \]
Then defining
\[ K_0^*(\xi, z) = M(\xi, z) / \int_B M(\xi, w) dv_w, \]
we have
\[ \frac{K_0^*(\xi, z)}{K_0^*(\xi, \xi)} = \frac{M(\xi, z)}{\int_B M(\xi, w) dv_w} \cdot \int_B M(\xi, w) dv_w = M(\xi, z). \]
In analogy to the case of one complex variable we call the function \( K_0(\xi, z) \) the Szegö kernel of the domain \( D \).

**Theorem 2.** Let \( M(\xi, z) \) be a solution of the above minimal problem and \( K_0(\xi, z) \) the kernel defined by (11). Then for any \( f \) regular on \( \overline{D} \) the (reproducing) formula
\[ f(\xi) = \int_B K_0(\xi, z)f(z) dv_z, \quad \xi \in D. \]
holds. Also the minimum value of the integral is \( \left[ 1/K_0(\xi, \xi) \right] \).
Proof. From the minimal property for any arbitrary complex \( \epsilon \) and regular \( f \)

\[
\int_B |M|^2 \, dv \leq \int_B |M + \epsilon (f(\zeta) - f(\zeta))|^2 \, dv.
\]

Thus

\[
2 \text{Re} \left[ \epsilon \int_B M^* (f(z) - f(\zeta)) \, dv \right] + |\epsilon|^2 \int_B |f(z) - f(\zeta)|^2 \, dv \geq 0.
\]

Since \( |\epsilon| \) and \( \text{arg} \, \epsilon \) are both arbitrary, it follows that

\[
\int_B M^*(\xi, z) [f(z) - f(\xi)] \, dv = 0,
\]

from which (12) results. Also from (12)

\[
\int_B \left| M(\xi, z) \right|^2 \, dv = \left| K^{-2}(\xi, \xi) \right| \int_B K_0(\xi, z) K_0^*(\xi, z) \, dv
\]

\[
= K_0^{-1}(\xi, \xi).
\]

For the matric spaces as we have seen in §2 this formula is valid for any \( f \) regular on \( D \) and continuous on \( \overline{D} \).

For the domains \( \mathbb{A}_n, \mathbb{E}_n \) and \( \mathbb{E}_m \) the kernel \( K_0(\xi, z) \) is equal to

(13)

\[
K_0(\xi, z) = V^{-1} \det^{-p} (I - z^* \xi),
\]

which equals \( \det^{2n} K(\xi, z) \) if \( z \in B \), where \( p \) has the same values as for the kernel \( K(\xi, z) \); for \( \mathbb{M}_n \)

(13a)

\[
K_0(\xi, z) = V^{-1} [1 - 2z^* \xi + (\xi^* \xi)(z'z)^*]^{-n/2}.
\]

The proof that the minimal problem has a solution for the matric spaces and that (13) satisfies (11) is similar to that in [6] for the Bergman kernel function and will be omitted here.

References

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A COUNTABLE INTERPOLATION PROBLEM

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1. Let \( H \) be the set of all order-preserving homeomorphisms of \( I = [0, 1] \) onto itself. \( H \) is a metric space in the uniform metric \( \rho \):

\[
\rho(f_1, f_2) = \max_{I} |f_1(x) - f_2(x)|, \quad f_1, f_2 \in H.
\]

Franklin [1] has proved the following theorem: (A) Let \( A \) and \( B \) be two countable sets, each dense on \( I \). Then the set of analytic \( f \in H \), such that \( f(A) = B \), is dense in \( H \).

It follows from (A) and from its extension in [2] that there exist nontrivial analytic functions \( f \in H \), such that \( f(x) \) is transcendental for each transcendental \( x \in I \), and for each algebraic \( x \in I \), \( x \) and \( f(x) \) are algebraic and of the same degree.

In this note, without using either of these results, we prove a similar but complementary statement by means of Baire’s Category Theorem.

**Theorem 1.** Let \( H_\alpha, \alpha > 2 \), be the subset of \( H \) consisting of all functions \( f \in H \), whose values are either rational or transcendental and approximable to degree \( > \alpha \), for each algebraic \( x \in I \). Then \( H_\alpha \) is a dense \( G_\delta \)-set of second category in \( H \).

2. Since \( H \) is not complete in \( \rho \), we first remetrize it. Let

\[
\sigma(f_1, f_2) = \rho(f_1, f_2) + \rho(f_1^{-1}, f_2^{-1}), \quad f_1, f_2 \in H.
\]

**Lemma 1.** \( H \) is complete in the \( \sigma \)-metric.

Let \( \mathcal{F} = I^I \) be the set of all continuous maps from \( I \) into \( I \), then \( \mathcal{F} \) is complete in \( \rho \). Let \( \{f_n\}, n = 1, 2, \ldots \), be a \( \sigma \)-Cauchy sequence in \( H \). Then \( \{f_n\} \) is also a \( \rho \)-Cauchy sequence in \( \mathcal{F} \), therefore \( f_n \rightarrow f \),

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