COMPLETE IDEALS IN LOCAL RINGS

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1. Let $L$ be a noetherian local ring with maximal ideal $M$. To each ideal $H$ of $L$ there corresponds an ideal $H_a$, the integral closure or $v$-completion of $H$, which consists of those elements $x$ in $L$ that depend integrally upon $H$ in the sense that an equation, $x^n + a_1x^{n-1} + \cdots + a_n = 0$, in which $a_i \in H^i$, $i = 1, 2, \ldots, n$, holds. Any ideal $H$ such that $H_a = H$ is called a $v$-complete ideal. It is well known [3] that $(H_a)_a = H_a$ for all ideals $H$. In this note we study the sequence \{ $(Q_n)_a = (M^n)_a$ \} of the $v$-completions of the powers of the maximal ideal $M$ in the case where $L$ is a two dimensional local domain that is integrally closed in its quotient field $F$. Subject to certain finiteness restrictions explained below it is shown that there is an ideal $X$ generated by a set of parameters and an integer $s$ such that for each positive integer $t$, $Q_{s+t} = Q_sX^t$. By means of this result it is shown that the length $\lambda(Q_n)$ of the $M$-primary ideal $Q_n$ is given by a polynomial in $n$ when $n$ is large. Finally, it is shown that if $L$ is the local ring at a point $P$ of a normal algebraic surface $\mathcal{V}$, if $\mathcal{U}$ is the transform of $\mathcal{V}$ defined by a quadratic transformation with center $P$, and if $\mathcal{U}_s$ is a derived normal model of $\mathcal{U}$ associated with the character of homogeneity $\delta$, then $\chi(\mathcal{V}, 2n\delta) - \chi(\mathcal{U}_s, n) = \lambda(Q_n)$, where $\chi$ is the Hilbert characteristic function. In particular, the constant term of the polynomial $\lambda(Q_n)$ measures the change in the relative arithmetic genus of $\mathcal{U}$ that is induced by the quadratic transformation.

Some of the ideas and techniques used below are related to those employed by Rees in his study of pseudo-valuations [6, and related papers]. They go back ultimately, however, to Zariski’s work on the reduction of singularities.

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2. A subset $S$ of an ideal $H$ in a noetherian local ring $(L, M)$ will be called an $a$-basis of $H$ in case $H_a = (LS)_a$. The ideal $LS$ is then a reduction of $H$ in the sense of Northcott-Rees who have shown [5] that if $\{x_1, x_2, \ldots, x_r\}$ is a minimal $a$-basis of the ideal $M$, then the number $r$ of the elements $x_i$ is equal to the dimension of $L$, and

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these elements are analytically independent. If $L$ is a domain integrally closed in its quotient field $F$ and if $\Omega$ is the set of valuations $v$ of $F$ such that $R_v$, the valuation ring of $v$, is an overring of $L$, and if $H_b = \{ x; v(x) \geq v(H), \forall v \in \Omega \}$, then $H_b = H_a$ [3; 7]. (Here $v(H)$ denotes $\min \{ v(a); a \in H \}$. ) Moreover, for each $v \in \Omega$ there is at least one $i$ such that $v(x_i) = v(M)$, so that $v(x_i^n) = v(M^n)$ for all integers $n \geq 0$. Hence if $X = \sum (Lx_i)$ is the ideal generated by the minimal $a$-basis $\{ x_1, x_2, \ldots , x_r \}$, then $v(X^n) = v(M^n)$, and $Q_n = (X^n)_b$.

Let $L_i = L[x_i/x_i, x_2/x_i, \ldots , x_r/x_i]$, and let $T_i$ be the integral closure of $L_i$ in $F$. Since the condition, $v(x_i) = v(M)$, implies that $R_v \supseteq L_i$ it follows that $T_i = \cap \{ R_v; v \in \Omega, v(x_i) = v(M) \}$, and $L = \cap T_i$. If $x \in Q_n$, then $x$ satisfies an equation, $x^t + a_1 x^{t-1} + \cdots + a_t = 0$, $a_i \in X_i^n$, so that $x/x_i^n \in T_j$, $j = 1, 2, \ldots , r$, and hence $Q_n \subseteq x_i^n T_j \cap L$. The minimal property of the $a$-basis $\{ x_1, x_2, \ldots , x_r \}$ assures us that $x_i$ is not a unit in $T_j$, and since $T_j$ is integral over $L_j$ there exists at least one real discrete valuation $v$ such that $R_v \supseteq T_j$ and $v(x_i) > 0$. Hence no element different from zero in $T_j$ is divisible by all powers of $x_i$ so that we have

$$\bigcap_{n=1}^{\infty} Q_n = (0).$$

We now assume that $L$ is complete and that its characteristic coincides with that of its residue field. By a theorem of Cohen [2], $L$ then contains a coefficient field $k$, and the power series ring

$$P = k\{ x_1, x_2, \ldots , x_r \}$$

is a subring of $L$ over which $L$ is a finite module. If

$$P_j = P[x_1/x_j, x_2/x_j, \ldots , x_r/x_j],$$

then $T_j$ is integral over $P_j$ and the relation between the rings $P$ and $P_j$ is similar to the relation between $L$ and $T_j$. If $X_0 = \sum P x_i$, and if $\theta \in X_0^n$, then $\theta/x_j^n \in P_j$. On the other hand if $\pi \in P_j$ then $\pi$ is a polynomial in the quotients $x_i/x_j$ so that if $n$ is its total degree, $x_j^n \pi$ is a form of degree $n$ in $x_1, x_2, \ldots , x_r$ with coefficients in $P$. Thus $x_j^n \pi \in X_0^n$. Moreover, since $P$ is a regular local ring the equation, $X_0^n \cdot (x_j^n) = X_0^n x_j^{-1}$, holds for all $n \geq t$, so that if $e$ is the least integer such that $x_j^n \pi \in P$, then $x_j^n \pi \in X_0^n$. This integer $e$ will be called the degree of $\pi$. If $y$ is an element of the quotient field of $P$ that is integral over $P_j$, there is an integer $e$ such that $x_j^e y \in (X_0^n)_b$. In fact, if $a_i$ is the coefficient of $y^{t-i}$ in the equation of integral dependence and if $e_i$ is the degree of $a_i$, we can take $e$ to be the least integer $f$ such that
However, the regularity of $P$ implies that the pseudo-valuation associated with the sequence of powers of $X_0$ is a valuation and that $X_0^n$ is a complete ideal for all positive integers $n$. Hence $x_0^ny \in X_0^n$ and $y \in P_j$. Thus $P_j$ is integrally closed.

**Lemma 1.** If $\theta \in Q_n$ and if $g(Z) = Z^t + a_t Z^{t-1} + \cdots + a_1$ is the minimal polynomial for $\theta$ over the quotient field $E$ of $P$, then $a_i \in X_0^n$.

**Proof.** As mentioned above, $\theta/x_0^n \in T_j$ and hence $\theta/x_0^n$ is integral over $P_j$. If $h(Z) = Z^t + b_t Z^{t-1} + \cdots + b_1$ is the minimal polynomial of $\theta/x_0^n$ over $E$, then by a well known lemma of Kronecker [3], $b_i \in P_j$, $i = 1, 2, \cdots, t$. For the same reason, $a_i \in P$. Hence since $a_i = b_i x_0^n$, it follows that $a_i \in X_0^n$, q.e.d.

**Lemma 2.** If $\theta \in T_j$ then there exist integers $s$ such that $x_0^s \theta \in L$. For any such integer $s$ it is true that $x_0^s \theta \in Q_s$. Moreover, for all positive integers $n$, $x_0^s T_j \cap L = Q_n$.

**Proof.** As above, the coefficients $a_i$ of the minimal polynomial $g(Z)$ of $\theta$ are in $P_j$, so that if $d_i$ is the degree of $a_i$ and $s$ is an integer such that $si \geq d_i$, $i = 1, 2, \cdots, t$ then the minimal equation of $x_0^s \theta$ has coefficients $x_0^s a_i$ in $P$, so that $x_0^s \theta \in L$. On the other hand if $s$ is such that $x_0^s \theta \in L$, then $x_0^s a_i \in P$ and hence $x_0^s a_i \in X_0^n$. Thus $x_0^s \theta$ depends integrally on $X^n$ so that $x_0^s \theta \in Q_s$. In particular, for any positive integer $n$ the inclusion, $x_0^s T_j \cap L \subseteq Q_n$, holds, and this establishes equality since the opposite inclusion has already been noted, q.e.d.

**Proposition 1.** Let $(L, M)$ be an integrally closed local domain of the same characteristic as its residue field which is such that its completion $(\overline{L}, \overline{M})$ is also an integrally closed domain. Let $k$ be a coefficient field of $L$ and let $k = k \cap L$. If $\phi(Z_1, Z_2, \cdots, Z_t)$ is a nonzero form of degree $s$ in the indeterminates $Z$ with coefficients in $k$, then for each positive integer $n \geq s$, the equality,

$$Q_n: (\phi(x_1, x_2, \cdots, x_t)) = Q_{n-s},$$

holds.

**Proof.** If $\overline{Q}_n = (\overline{M}^n)_s$, it follows immediately that $\overline{Q}_n = L Q_n$, so that it is sufficient to prove the proposition under the assumption that $L$ is complete. Assume that $\phi$ is as described and that $\phi(x) \theta \in Q_n$. If $g(Z) = Z^t + a_t Z^{t-1} + \cdots + a_1$ is the minimal polynomial for $\theta$ over the ring $P$, then $Z^t + \phi a_t Z^{t-1} + \cdots + \phi a_1$ is the minimal polynomial of $\phi(x) \theta$ over $P$, and by Lemma 1, $(\phi(x)) a_i \in X_0^n$. Since $P$ is a regular local ring, it follows that $a_i \in X_0^{n-s}$ so that $\theta \in Q_{n-s}$, q.e.d.
3. Let $L$ be a two dimensional local domain that is integrally closed in its quotient field $F$, and let the notations of the previous sections be retained.

**Lemma 3.** If $g$ and $h$ are positive integers, then

$$(X^gQ_h): (Lx_j) = X^{g-1}Q_h, \quad j = 1, 2.$$  

**Proof.** If $x_i\theta \in X^gQ_h$, then

$$(x_i\theta) = \alpha_0x_1^g + \alpha_1x_1^{g-1}x_2 + \cdots + \alpha_gx_2^g,$$

where $\alpha_i \in Q_h, i = 0, 1, \ldots, g$. Since $L$ is integrally closed, the principal ideal $Lx_1$ has no embedded components and hence $x_2$ is in no prime ideal of $Lx_1$. Hence equation (2) implies that $\alpha_0 = x_1\beta$, and by Proposition 1, $\beta \in Q_{h-1}, x_2\beta \in Q_h$. Hence we can replace the term $\alpha_0x_2^g$ in (2) by $x_1\beta x_2^g$ to obtain $\theta \in X^{g-1}Q_h$, q.e.d.

We now formulate two conditions either of which will insure the finiteness of the integral closure of a ring which occurs in the proof of Proposition 2. The first of these is similar to a condition imposed by Rees in [6]. An integrally closed local domain $L$ will be said to satisfy condition $f_1$ in case

(a) $L$ has the same characteristic as its residue field;
(b) The completion $\hat{L}$ of $L$ is an integral domain with quotient field $\hat{F}$;
(c) The integral closure $T$ in $\hat{F}$ of any finite ring extension $S$ of $L$ such that $S \subseteq \hat{F}$ is a finite $S$-module.

An alternative requirement $f_2$ would also serve our purpose. In view of (a) and a theorem of Cohen [2], the ring $\hat{L}$ is a finite module over the power series ring $P = k\{x_1, x_2, \ldots, x_r\}$, where $k$ is a coefficient field for $L$ and $x_1, x_2, \ldots, x_r$ are parameters. Hence the quotient field $F$ of $\hat{L}$ is a finite algebraic extension of the quotient field $E$ of $P$. We shall say that $L$ satisfies condition $f_2$ if (a) and (b) hold and if $F$ is separable over $E$.

**Proposition 2.** If $(L, M)$ is a two dimensional integrally closed local domain that satisfies either condition $f_1$ or condition $f_2$, and if $X$ is the ideal generated by a minimal $a$-basis $x_1, x_2$ of $M$, then there exists an integer $s$ such that $Q_{s+t} = X^tQ_s$ for all positive integers $t$.

**Proof.** It is clearly sufficient to prove the proposition under the assumption that $L$ is itself complete. If $T_j, L_j$ and $P_j (j = 1, 2)$ are as defined in §2, then condition $f_1$ asserts that $T_j$ is a finite $L_j$-module. This conclusion can also be deduced from condition $f_2$. In fact, $L_j$ is integral over $P_j$ so that $T_j$ is the integral closure in $F$ (the quotient field of the complete domain $L_j$) of the noetherian domain $P_j$, and
since $F$ is separable over $E$ the conclusion follows from the fact that $P_j$ is integrally closed.

Let $\omega_1, \omega_2, \ldots, \omega_m$ be a basis for $T_1$ as an $L_1$ module. By Lemma 2 there is an integer $s$ such that $x_i^s\omega_i \in L$, $i = 1, 2, \ldots, m$, and for such an integer $s$, $x_i^s\omega_i \in Q_s$. Now assume that $\theta \in Q_n$, $n \geq s$, so that $\theta/x_i^s \in T_1$. There are elements $u_1, u_2, \ldots, u_m$ of $L_1$ such that

$$\theta/x_1^n = u_1\omega_1 + u_2\omega_2 + \cdots + u_m\omega_m.\quad(3)$$

Since $u_i$ is a polynomial in $x_2/x_1$ with coefficients in $L$ it follows that there is an integer $g$ such that $x_i^g u_i \in X^g$, $i = 1, 2, \ldots, m$. Hence if $\mu = \max (s+g, n)$ we can multiply equation (3) by $x_i\mu$ to get $x_i^{-n}\theta \in X^{s-g}Q_s$. If $\mu = n$, then $\theta \in X^{s-g}Q_s$, and the proof is complete. Otherwise, $\mu = s+g$ and we have $x_i^{s+g} \theta \in X^gQ_s$, so that by repeated application of Lemma 3 we again find $\theta \in X^{s-g}Q_s$, q.e.d.

**Corollary.** For all integers $a, b \geq s$, $Q_aQ_b = Q_{a+b}$. In particular, $Q_n^s = Q_{ns}$ for all positive integers $n$.

**Proof.** $Q_{a+b} \supseteq Q_aQ_b \supseteq X^{a+b-2s}Q_s \supseteq X^{a+b-s}Q_s = Q_{a+b}$.

**Proposition 3.** Under the same hypothesis as Proposition 2, there is an integer $n_0$ such that when $n \geq n_0$ the length $\lambda(Q_n)$ of $Q_n$ is given by a polynomial in $n$ of degree two that has the same leading term as the Samuel polynomial $s(n)$ that gives the length of $X^n$ when $n$ is large.

**Proof.** It is known [1] that there exist integers $\alpha$ and $\beta$ such that for $a \geq \alpha$ and $b \geq \beta$, $\lambda(X^nQ_\alpha) = B(a, b)$, where $B(a, b)$ is a polynomial in $a, b$ of total degree equal to the dimension of $L$. Hence if $n_0 = s\beta + \alpha$, and if $n \geq n_0$, Proposition 2 implies that $Q_n = X^{n-s\beta}Q_\beta$, so that $\lambda(Q_n) = B(n - s\beta, \beta)$. The conclusion concerning the leading coefficients follows from the fact that $X^{n-s} \supseteq Q_n \supseteq X^n$, so that $\lambda(X^{n-s}) \leq \lambda(Q_n) \leq \lambda(X^n)$, q.e.d.²

4. Let $(x_0, x_1, \ldots, x_m)$ be homogeneous coordinates of a general point of an arithmetically normal surface $V$ defined over an algebraically closed ground field $k$, assume that the point $(1, 0, \ldots, 0)$ is on $V$, that $x_0, x_1, x_2$ are algebraically independent over $k$ and that the coordinate ring $R = k[x_0, x_1, \ldots, x_m]$ is integral over the ring $k[x_0, x_1, x_2]$. If $y_i = x_i/x_0$, then $\Sigma = k(y_1, y_2, \ldots, y_m)$ is the field of rational functions on $V$. Let $z_{ij} = x_ix_j$, $i, j = 0, 1, \ldots, m$. The ring $R_2 = k[z_{00}, z_{01}, \ldots, z_{mm}]$ is the ring of homogeneous coordinates on a

² The referee has called attention to the fact that in view of Proposition 2, the graded ring $\Sigma Q_n/\Sigma Q_{n+1}$ (in the sense of [8, Chapter II]) is a finite module over the ring $\Sigma X^n/\Sigma X^{n+1}$, and that Proposition 3 follows from this.
derived normal model \( \mathcal{U}_2 \) of \( \mathcal{U} \) belonging to the character of homogeneity 2 [9], while the ring \( S = k [z_{01}, z_{02}, \ldots, z_{mn}] \) is the ring of homogeneous coordinates on the surface \( \mathcal{U} \) obtained from \( \mathcal{U} \) by a quadratic transformation with center \( P = (1, 0, \ldots, 0) \). Since \( z_{00} = z_{01} z_{02} / z_{11} \) it follows that \( \Sigma(z_{00}) = \Sigma(z_{01}) \) so that \( R_2 \) and \( S \) have the same quotient field. Hence if \( S_1 \) is the ring of homogeneous coordinates on a derived arithmetically normal model \( \mathcal{U}_1 \) of \( \mathcal{U} \) belonging to the character of homogeneity \( \delta \), then \( S_1 \) is a subring of \( R_2 \) in view of the fact that \( R_2 \) is integrally closed. An element of \( S_2 \) which is homogeneous of degree \( n \) in the natural grading of \( S_1 \) will be of degree \( n \delta \) if regarded as an element of \( R_2 \) and of degree \( 2n \delta \) as an element of \( R \), so that the space \( U(n) \) of forms in \( S_1 \) of degree \( n \) is a subspace of the space \( V(2n \delta) \) of forms in \( R \) of degree \( 2n \delta \).

If \( \mathcal{O} = k [y_1, y_2, \ldots, y_m] \), \( \mathcal{O} \mathcal{N} = \sum y_i \) and if \( \mathcal{N}_1 \) is the complete ideal \( (\mathcal{N}_1)_0 \), then it is clear that \( \mathcal{N}_1 = Q_t \cap \mathcal{O} \), where \( Q_t \) is the integral closure of \( M^t \) in the local ring \( (\mathcal{O}_N, M) \) at the point \( P \) of \( \mathcal{U} \). In particular, \( Q_t \) and \( \mathcal{N}_1 \) have the same length.

**Lemma 4.** An element \( \omega \) of \( V(2n \delta) \) belongs to the space \( U(n) \) if and only if \( \omega / x_0^{2n \delta} \) is an element of the ideal \( N_{n \delta} \).

**Proof.** If \( \omega \in U(n) \) then \( \omega \) satisfies an equation of the form \( \omega^r + a_1 \omega^{r-1} + \cdots + a_s = 0 \), where \( a_i \) is an element of \( S \) of degree \( n \delta i \). It follows that \( a_i / x_0^{2n \delta} \) is an element of \( \omega \) in which each term is a power product in \( y_1, y_2, \ldots, y_m \) of degree not less than \( n \delta i \). Hence \( \omega / x_0^{2n \delta} \) is integral over \( N_{n \delta} \) and is therefore in \( N_{n \delta} \).

Assume on the other hand, that \( \omega \in V(2n \delta) \) and that \( \theta = \omega / x_0^{2n \delta} \subseteq N_{n \delta} \). The element \( \theta \) satisfies an equation of the form \( \theta^r + b_1 \theta^{r-1} + \cdots + b_t = 0 \), with \( b_i \in N_{n \delta i} \). Thus the coefficient \( b_i \) can be written as a polynomial in \( y_1, y_2, \ldots, y_m \) in which no term is of degree less than \( n \delta i \). In view of the relations, \( z_{00} y_k = z_{0k}, z_{0ij} y_k = z_{jk} \), it follows that \( z_{00}^{n \delta i} b_i \) is an element of \( S \). Hence if we multiply the equation for \( \theta \) by \( z_{00}^{n \delta i} \) we find an equation that expresses the integral dependence of \( \omega \) on \( S \). Since \( \delta \) is a character of homogeneity, it follows that \( \omega \) is an element of \( S_1 \) of degree \( n \), q.e.d.

Since the dimensions of the spaces \( U(n) \) and \( V(2n \delta) \) are given by the Hilbert functions \( \chi(\mathcal{U}_1, n) \) and \( \chi(\mathcal{U}, 2n \delta) \) respectively, and since \( \lambda(N_n) = \lambda(Q_n) \) for all \( n \), the equality,

\[
(4) \quad \chi(\mathcal{U}, 2n \delta) - \chi(\mathcal{U}_1, n) = \lambda(Q_{n \delta}),
\]

is an immediate consequence of Lemma 4.

It should be noted that equation (4) implies that the length \( \lambda(Q_{n \delta}) \) is a polynomial in \( n \) when \( n \) is large, so that if the local ring \((\mathcal{O}_N, M)\)
satisfies either condition $f_1$ or $f_3$, this polynomial will coincide with
the one described in Proposition 3 with $n$ replaced by $n\delta$. It is not
difficult to see that $(0_N, M)$ satisfies condition $f_2$. In fact, condition
(a) is automatically satisfied and condition (b) is a consequence of
the analytical irreducibility of normal varieties. As to the separabil-
ity requirement imposed by $f_2$ we note first that since $k$ is algebraically
closed we can assume without loss of generality that the quantities
$(y_1, y_2)$ form a separating transcendence base for $\Sigma/k$ and at the same
time form a minimal $a$-basis for the maximal ideal $M$ in $0_N$. If $(\overline{L}, \overline{M})$
is the completion of $0_N$ relative to the powers of $M$, and if $P = k\{y_1, y_2\}$,
then $\overline{L}$ is integral over $P$ and it is a straightforward matter to see
that every $k$-derivation of $P$ admits a unique extension to a $k$-deriva-
tion of $\overline{L}$. Indeed, the term "$k$-derivation" is used here in the sense of
[4, §3], and such derivations are uniquely determined by their re-
strictions to dense subrings. Since $0_N$ is dense in $\overline{L}$ and the poly-
nomial ring $P_0 = k[y_1, y_2]$ is dense in $P$, and since any $k$-derivation of
$P_0$ admits a unique extension to $0_N$ in view of the separability of $\Sigma$
over $k(y_1, y_2)$, our assertion follows. Hence it follows also that the
quotient field $F$ of $L$ is separable over the quotient field $E$ of $P$ as
condition $f_2$ requires.

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