COMPLETE IDEALS IN LOCAL RINGS[1]

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1. Let L be a noetherian local ring with maximal ideal M. To each ideal H of L there corresponds an ideal H_a, the integral closure or v-completion of H, which consists of those elements x in L that depend integrally upon H in the sense that an equation, $x^n+a_1x^{n-1}+\cdots+a_n=0$, in which $a_i\in H^i$, $i=1, 2, \cdots, n$, holds. Any ideal H such that $H_a=H$ is called a v-complete ideal. It is well known [3] that $(H_a)_a=H_a$ for all ideals H. In this note we study the sequence $\{Q_n\}$ $(Q_n=(M^n)_a)$ of the v-completions of the powers of the maximal ideal M in the case where L is a two dimensional local domain that is integrally closed in its quotient field F. Subject to certain finiteness restrictions explained below it is shown that there is an ideal X generated by a set of parameters and an integer s such that for each positive integer t, $Q_{s+t}=Q_sX^t$. By means of this result it is shown that the length $\lambda(Q_n)$ of the M-primary ideal Q_n is given by a polynomial in n when n is large. Finally, it is shown that if L is the local ring at a point P of a normal algebraic surface $\mathfrak{V}$, if $\mathfrak{U}$ is the transform of $\mathfrak{V}$ defined by a quadratic transformation with center P, and if $\mathfrak{U}_\delta$ is a derived normal model of $\mathfrak{U}$ associated with the character of homogeneity δ, then $\chi(\mathfrak{V}, 2n\delta)-\chi(\mathfrak{U}_\delta, n)=\lambda(Q_{n\delta})$, where χ is the Hilbert characteristic function. In particular, the constant term of the polynomial $\lambda(Q_n)$ measures the change in the relative arithmetic genus of $\mathfrak{V}$ that is induced by the quadratic transformation.

Some of the ideas and techniques used below are related to those employed by Rees in his study of pseudo-valuations [6, and related papers]. They go back ultimately, however, to Zariski's work on the reduction of singularities.

The author expresses his thanks to the referee for several helpful suggestions.

2. A subset S of an ideal H in a noetherian local ring (L, M) will be called an a-basis of H in case $H_a=(LS)_a$. The ideal LS is then a *reduction* of H in the sense of Northcott-Rees who have shown [5] that if $\{x_1, x_2, \cdots, x_r\}$ is a minimal a-basis of the ideal M, then the number r of the elements x_i is equal to the dimension of L, and

Presented to the Society, November 23, 1956 under the title *On the relative arithmetic genus of a normal surface*; received by the editors August 10, 1959.

[1] Work on this paper was supported in part by a grant from the National Science Foundation.

these elements are analytically independent. If L is a domain integrally closed in its quotient field F and if Ω is the set of valuations v of F such that R_v, the valuation ring of v, is an overring of L, and if $H_b = \{x; v(x) \geqq v(H), \forall v \in \Omega\}$, then $H_b = H_a$ [3; 7]. (Here $v(H)$ denotes $\min\{v(a); a \in H\}$.) Moreover, for each $v \in \Omega$ there is at least one i such that $v(x_i) = v(M)$, so that $v(x_i^n) = v(M^n)$ for all integers $n \geqq 0$. Hence if $X = \sum(Lx_i)$ is the ideal generated by the minimal a-basis $\{x_1, x_2, \cdots, x_r\}$, then $v(X^n) = v(M^n)$, and $Q_n = (X^n)_b$.

Let $L_i = L[x_1/x_i, x_2/x_i, \cdots, x_r/x_i]$, and let T_i be the integral closure of L_i in F. Since the condition, $v(x_i) = v(M)$, implies that $R_v \supseteq L_i$ it follows that $T_i = \bigcap\{R_v; v \in \Omega, v(x_i) = v(M)\}$, and $L = \bigcap T_i$. If $x \in Q_n$, then x satisfies an equation, $x^t + a_1 x^{t-1} + \cdots + a_t = 0$, $a_i \in X^{ni}$, so that $x/x_j^n \in T_j$, $j = 1, 2, \cdots, r$, and hence $Q_n \subseteq x_j^n T_j \cap L$. The minimal property of the a-basis $\{x_1, x_2, \cdots, x_r\}$ assures us that x_j is not a unit in T_j, and since T_j is integral over L_j there exists at least one real discrete valuation v such that $R_v \supseteq T_j$ and $v(x_j) > 0$. Hence no element different from zero in T_j is divisible by all powers of x_j so that we have

$$\bigcap_{n=1}^{\infty} Q_n = (0). \tag{1}$$

We now assume that L is complete and that its characteristic coincides with that of its residue field. By a theorem of Cohen [2], L then contains a coefficient field k, and the power series ring

$$P = k\{x_1, x_2, \cdots, x_r\}$$

is a subring of L over which L is a finite module. If

$$P_j = P[x_1/x_j, x_2/x_j, \cdots, x_r/x_j],$$

then T_j is integral over P_j and the relation between the rings P and P_j is similar to the relation between L and T_j. If $X_0 = \sum Px_i$, and if $\theta \in X_0^n$, then $\theta/x_j^n \in P_j$. On the other hand if $\pi \in P_j$ then π is a polynomial in the quotients x_i/x_j so that if n is its total degree, $x_j^n \pi$ is a form of degree n in $x_1, x_2, \cdots, x_r$ with coefficients in P. Thus $x_j^n \pi \in X_0^n$. Moreover, since P is a regular local ring the equation, $X_0^n : (x_j^t) = X_0^{n-t}$, holds for all $n \geqq t$, so that if e is the least integer such that $x_j^e \pi \in P$, then $x_j^e \pi \in X_0^e$. This integer e will be called the degree of π. If y is an element of the quotient field of P that is integral over P_j, there is an integer e such that $x_j^e y \in (X_0^e)_b$. In fact, if a_i is the coefficient of y^{t-i} in the equation of integral dependence and if e_i is the degree of a_i, we can take e to be the least integer f such that

$fi \geqq e_i$, $i=1, 2, \cdots, t$. However, the regularity of P implies that the pseudo-valuation associated with the sequence of powers of X_0 is a valuation and that X_0^n is a complete ideal for all positive integers n. Hence $x_j^e y \in X_0^e$ and $y \in P_j$. Thus P_j is integrally closed.

LEMMA 1. *If $\theta \in Q_n$ and if $g(Z)=Z^t+a_1Z^{t-1}+\cdots+a_t$ is the minimal polynomial for θ over the quotient field E of P, then $a_i \in X_0^{ni}$.*

PROOF. As mentioned above, $\theta/x_j^n \in T_j$ and hence θ/x_j^n is integral over P_j. If $h(Z)=Z^t+b_1Z^{t-1}+\cdots+b_t$ is the minimal polynomial of θ/x_j^n over E, then by a well known lemma of Kronecker [3], $b_i \in P_j$, $i=1, 2, \cdots, t$. For the same reason, $a_i \in P$. Hence since $a_i = b_i x_j^{ni}$, it follows that $a_i \in X_0^{ni}$, q.e.d.

LEMMA 2. *If $\theta \in T_j$ then there exist integers s such that $x_j^s\theta \in L$. For any such integer s it is true that $x_j^s\theta \in Q_s$. Moreover, for all positive integers n, $x_j^n T_j \cap L = Q_n$.*

PROOF. As above, the coefficients a_i of the minimal polynomial $g(Z)$ of θ are in P_j, so that if d_i is the degree of a_i and s is an integer such that $si \geqq d_i$, $i=1, 2, \cdots, t$ then the minimal equation of $x_j^s\theta$ has coefficients $x_j^{si}a_i$ in P, so that $x_j^s\theta \in L$. On the other hand if s is such that $x_j^s\theta \in L$, then $x_j^{si}a_i \in P$ and hence $x_j^{si}a_i \in X_0^{si}$. Thus $x_j^s\theta$ depends integrally on X^s so that $x_j^s\theta \in Q_s$. In particular, for any positive integer n the inclusion, $x_j^n T_j \cap L \subseteq Q_n$, holds, and this establishes equality since the opposite inclusion has already been noted, q.e.d.

PROPOSITION 1. *Let (L, M) be an integrally closed local domain of the same characteristic as its residue field which is such that its completion $(\bar{L}, \bar{M})$ is also an integrally closed domain. Let $\bar{k}$ be a coefficient field of $\bar{L}$ and let $k=\bar{k}\cap L$. If $\phi(Z_1, Z_2, \cdots, Z_r)$ is a nonzero form of degree s in the indeterminates Z with coefficients in k, then for each positive integer $n \geqq s$, the equality,*

$$Q_n\colon (\phi(x_1, x_2, \cdots, x_r)) = Q_{n-s},$$

holds.

PROOF. If $\bar{Q}_n = (\bar{M}^n)_a$ it follows immediately that $\bar{Q}_n = \bar{L}Q_n$, so that it is sufficient to prove the proposition under the assumption that L is complete. Assume that ϕ is as described and that $\phi(x)\theta \in Q_n$. If $g(Z)=Z^t+a_1Z^{t-1}+\cdots+a_t$ is the minimal polynomial for θ over the ring P, then $Z^t+\phi a_1 Z^{t-1}+\cdots+\phi^t a_t$ is the minimal polynomial of $\phi(x)\theta$ over P, and by Lemma 1, $(\phi(x))^i a_i \in X_0^{ni}$. Since P is a regular local ring, it follows that $a_i \in X_0^{ni-si}$ so that $\theta \in Q_{n-s}$, q.e.d.

3. Let L be a two dimensional local domain that is integrally closed in its quotient field F, and let the notations of the previous sections be retained.

LEMMA 3. *If g and h are positive integers, then*

$$(X^gQ_h):(Lx_j) = X^{g-1}Q_h, \qquad j = 1, 2.$$

PROOF. If $x_1\theta \in X^gQ_h$, then

$$x_1\theta = \alpha_0x_1^g + \alpha_1x_1^{g-1}x_2 + \cdots + \alpha_gx_2^g, \tag{2}$$

where $\alpha_i \in Q_h$, $i=0, 1, \cdots, g$. Since L is integrally closed, the principal ideal Lx_1 has no embedded components and hence x_2 is in no prime ideal of Lx_1. Hence equation (2) implies that $\alpha_g = x_1\beta$, and by Proposition 1, $\beta \in Q_{h-1}$, $x_2\beta \in Q_h$. Hence we can replace the term $\alpha_gx_2^g$ in (2) by $x_1\beta x_2^g$ to obtain $\theta \in X^{g-1}Q_h$, q.e.d.

We now formulate two conditions either of which will insure the finiteness of the integral closure of a ring which occurs in the proof of Proposition 2. The first of these is similar to a condition imposed by Rees in [6]. An integrally closed local domain L will be said to satisfy condition f_1 in case

(a) *L has the same characteristic as its residue field*;

(b) *The completion $\overline{L}$ of L is an integral domain with quotient field $\overline{F}$*;

(c) *The integral closure T in $\overline{F}$ of any finite ring extension S of $\overline{L}$ such that $S \subseteq \overline{F}$ is a finite S-module.*

An alternative requirement f_2 would also serve our purpose. In view of (a) and a theorem of Cohen [2], the ring $\overline{L}$ is a finite module over the power series ring $P = k\{x_1, x_2, \cdots, x_r\}$, where k is a coefficient field for $\overline{L}$ and $x_1, x_2, \cdots, x_r$ are parameters. Hence the quotient field $\overline{F}$ of $\overline{L}$ is a finite algebraic extension of the quotient field E of P. We shall say that L satisfies condition f_2 if (a) and (b) hold and if $\overline{F}$ is separable over E.

PROPOSITION 2. *If (L, M) is a two dimensional integrally closed local domain that satisfies either condition f_1 or condition f_2, and if X is the ideal generated by a minimal a-basis x_1, x_2 of M, then there exists an integer s such that $Q_{s+t} = X^tQ_s$ for all positive integers t.*

PROOF. It is clearly sufficient to prove the proposition under the assumption that L is itself complete. If T_j, L_j and P_j $(j=1, 2)$ are as defined in §2, then condition f_1 asserts that T_j is a finite L_j-module. This conclusion can also be deduced from condition f_2. In fact, L_j is integral over P_j so that T_j is the integral closure in F (the quotient field of the complete domain L) of the noetherian domain P_j, and

since F is separable over E the conclusion follows from the fact that P_j is integrally closed.

Let $\omega_1, \omega_2, \cdots, \omega_m$ be a basis for T_1 as an L_1 module. By Lemma 2 there is an integer s such that $x_1^s\omega_i \in L$, $i=1, 2, \cdots, m$, and for such an integer s, $x_1^s\omega_i \in Q_s$. Now assume that $\theta \in Q_n$, $n \geqq s$, so that $\theta/x_1^n \in T_1$. There are elements $u_1, u_2, \cdots, u_m$ of L_1 such that

$$\theta/x_1^n = u_1\omega_1 + u_2\omega_2 + \cdots + u_m\omega_m. \tag{3}$$

Since u_i is a polynomial in x_2/x_1 with coefficients in L it follows that there is an integer g such that $x_1^g u_i \in X^g$, $i=1, 2, \cdots, m$. Hence if $\mu = \max(s+g, n)$ we can multiply equation (3) by x_1^μ to get $x_1^{\mu-n}\theta \in X^{\mu-s}Q_s$. If $\mu = n$, then $\theta \in X^{n-s}Q_s$, and the proof is complete. Otherwise, $\mu = s+g$ and we have $x_1^{s+g-n}\theta \in X^g Q_s$, so that by repeated application of Lemma 3 we again find $\theta \in X^{n-s}Q_s$, q.e.d.

COROLLARY. *For all integers a, $b \geqq s$, $Q_aQ_b = Q_{a+b}$. In particular, $Q_s^n = Q_{ns}$ for all positive integers n.*

PROOF. $Q_{a+b} \supseteq Q_aQ_b \supseteq X^{a+b-2s}Q_s^2 \supseteq X^{a+b-s}Q_s = Q_{a+b}$.

PROPOSITION 3. *Under the same hypothesis as Proposition 2, there is an integer n_0 such that when $n \geqq n_0$ the length $\lambda(Q_n)$ of Q_n is given by a polynomial in n of degree two that has the same leading term as the Samuel polynomial $\sigma(n)$ that gives the length of X^n when n is large.*

PROOF. It is known [1] that there exist integers α and β such that for $a \geqq \alpha$ and $b \geqq \beta$, $\lambda(X^aQ_s^b) = B(a, b)$, where $B(a, b)$ is a polynomial in a, b of total degree equal to the dimension of L. Hence if $n_0 = s\beta + \alpha$, and if $n \geqq n_0$, Proposition 2 implies that $Q_n = X^{n-s\beta}Q_s^\beta$, so that $\lambda(Q_n) = B(n-s\beta, \beta)$. The conclusion concerning the leading coefficients follows from the fact that $X^{n-s} \supseteq Q_n \supseteq X^n$, so that $\lambda(X^{n-s}) \leqq \lambda(Q_n) \leqq \lambda(X^n)$, q.e.d.[2]

4. Let $(x_0, x_1, \cdots, x_m)$ be homogeneous coordinates of a general point of an arithmetically normal surface $\mathfrak{V}$ defined over an algebraically closed ground field k, assume that the point $(1, 0, \cdots, 0)$ is on $\mathfrak{V}$, that x_0, x_1, x_2 are algebraically independent over k and that the coordinate ring $R = k[x_0, x_1, \cdots, x_m]$ is integral over the ring $k[x_0, x_1, x_2]$. If $y_i = x_i/x_0$, then $\Sigma = k(y_1, y_2, \cdots, y_m)$ is the field of rational functions on $\mathfrak{V}$. Let $z_{ij} = x_ix_j$, $i, j = 0, 1, \cdots, m$. The ring $R_2 = k[z_{00}, z_{01}, \cdots, z_{mm}]$ is the ring of homogeneous coordinates on a

[2] The referee has called attention to the fact that in view of Proposition 2, the graded ring $\Sigma Q_n/Q_{n+1}$ (in the sense of [8, Chapter II]) is a finite module over the ring $\Sigma X^n/X^{n+1}$, and that Proposition 3 follows from this.

derived normal model $\mathfrak{V}_2$ of $\mathfrak{V}$ belonging to the character of homogeneity 2 [9], while the ring $S=k[z_{01}, z_{02}, \cdots, z_{mm}]$ is the ring of homogeneous coordinates on the surface $\mathfrak{U}$ obtained from $\mathfrak{V}$ by a quadratic transformation with center $P=(1, 0, \cdots, 0)$. Since $z_{00}=z_{01}z_{01}/z_{11}$ it follows that $\Sigma(z_{00})=\Sigma(z_{01})$ so that R_2 and S have the same quotient field. Hence if S_δ is the ring of homogeneous coordinates on a derived arithmetically normal model $\mathfrak{U}_\delta$ of $\mathfrak{U}$ belonging to the character of homogeneity δ, then S_δ is a subring of R_2 in view of the fact that R_2 is integrally closed. An element of S_δ which is homogeneous of degree n in the natural grading of S_δ will be of degree $n\delta$ if regarded as an element of R_2 and of degree $2n\delta$ as an element of R, so that the space $U(n)$ of forms in S_δ of degree n is a subspace of the space $V(2n\delta)$ of forms in R of degree $2n\delta$.

If $\mathfrak{o}=k[y_1, y_2, \cdots, y_m]$, if $N=\sum \mathfrak{o}y_i$ and if N_t is the complete ideal $(N^t)_a$, then it is clear that $N_t=Q_t\cap\mathfrak{o}$, where Q_t is the integral closure of M^t in the local ring $(\mathfrak{o}_N, M)$ at the point P of $\mathfrak{V}$. In particular, Q_t and N_t have the same length.

LEMMA 4. *An element ω of $V(2n\delta)$ belongs to the space $U(n)$ if and only if $\omega/x_0^{2n\delta}$ is an element of the ideal $N_{n\delta}$.*

PROOF. If $\omega\in U(n)$ then ω satisfies an equation of the form $\omega^s+a_1\omega^{s-1}+\cdots+a_s=0$, where a_i is an element of S of degree $n\delta i$. It follows that $a_i/x_0^{2n\delta i}$ is an element of $\mathfrak{o}$ in which each term is a power product in $y_1, y_2, \cdots, y_m$ of degree not less than $n\delta i$. Hence $\omega/x_0^{2n\delta}$ is integral over $N^{n\delta}$ and is therefore in $N_{n\delta}$.

Assume on the other hand, that $\omega\in V(2n\delta)$ and that $\theta=\omega/x_0^{2n\delta}\in N_{n\delta}$. The element θ satisfies an equation of the form $\theta^t+b_1\theta^{t-1}+\cdots+b_t=0$, with $b_i\in N^{n\delta i}$. Thus the coefficient b_i can be written as a polynomial in $y_1, y_2, \cdots, y_m$ in which no term is of degree less than $n\delta i$. In view of the relations, $z_{00}y_k=z_{0k}$, $z_{0j}y_k=z_{jk}$, it follows that $z_{00}^{n\delta t}b_i$ is an element of S. Hence if we multiply the equation for θ by $z_{00}^{n\delta t}$ we find an equation that expresses the integral dependence of ω on S. Since δ is a character of homogeneity, it follows that ω is an element of S_δ of degree n, q.e.d.

Since the dimensions of the spaces $U(n)$ and $V(2n\delta)$ are given by the Hilbert functions $\chi(\mathfrak{U}_\delta, n)$ and $\chi(\mathfrak{V}, 2n\delta)$ respectively, and since $\lambda(N_n)=\lambda(Q_n)$ for all n, the equality,

$$\chi(\mathfrak{V}, 2n\delta) - \chi(\mathfrak{U}_\delta, n) = \lambda(Q_{n\delta}), \tag{4}$$

is an immediate consequence of Lemma 4.

It should be noted that equation (4) implies that the length $\lambda(Q_{n\delta})$ is a polynomial in n when n is large, so that if the local ring $(\mathfrak{o}_N, M)$

satisfies either condition f_1 or f_2, this polynomial will coincide with the one described in Proposition 3 with n replaced by $n\delta$. It is not difficult to see that $(\mathfrak{o}_N, M)$ satisfies condition f_2. In fact, condition (a) is automatically satisfied and condition (b) is a consequence of the analytical irreducibility of normal varieties. As to the separability requirement imposed by f_2 we note first that since k is algebraically closed we can assume without loss of generality that the quantities (y_1, y_2) form a separating transcendence base for Σ/k and at the same time form a minimal a-basis for the maximal ideal M in $\mathfrak{o}_N$. If $(\overline{L}, \overline{M})$ is the completion of $\mathfrak{o}_N$ relative to the powers of M, and if $P = k\{y_1, y_2\}$, then $\overline{L}$ is integral over P and it is a straightforward matter to see that every k-derivation of P admits a unique extension to a k-derivation of $\overline{L}$. Indeed, the term "k-derivation" is used here in the sense of [4, §3], and such derivations are uniquely determined by their restrictions to dense subrings. Since $\mathfrak{o}_N$ is dense in $\overline{L}$ and the polynomial ring $P_0 = k[y_1, y_2]$ is dense in P, and since any k-derivation of P_0 admits a unique extension to $\mathfrak{o}_N$ in view of the separability of Σ over $k(y_1, y_2)$, our assertion follows. Hence it follows also that the quotient field F of $\overline{L}$ is separable over the quotient field E of P as condition f_2 requires.

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