CONCERNING STRONG LIE IDEALS

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Let A be a *simple ring of characteristic* $\neq 2$ *or* 3, *with either its center* $Z=(0)$ *or of dimension greater than* 16 *over its center, and with an involution defined on it.* Let S and K be the sets of symmetric and skew elements respectively. The Lie and Jordan products are $[u, v] = uv - vu$ and $u \circ v = uv + vu$. Denote by $[K, K]$ the additive subgroup of K generated by the elements $[k, m]$, k and m in K. An additive subgroup, U, of K is called a Lie ideal of K if $[u, k]$ is in U for all u in U, k in K.

Herstein [2] defines a Lie ideal, U, of K to be a *strong* Lie ideal if u in U implies u^3 is in U. In this paper we investigate the structure of strong Lie ideals in simple rings with involution. The main result of this paper is:

THEOREM 1. *If A is as above and U is a strong Lie ideal, then either U is contained in Z or $U=K$.*

In order to prove this theorem we make two remarks and prove several lemmas.

REMARK 1. $S^2 = S + [K, K]$. This follows from $S = S \circ S$ (cf. [2, Theorem 8]) and $[S, S] = [K, K]$ (cf. [1, Lemma 10]).

REMARK 2. There exists k in $[K, K]$ such that $k^2 \notin Z$. This is shown in the proof of Theorem 4 [1].

We use Remark 1 in the proof of the following lemma:

LEMMA 1. *If $s \in S$, $s \notin Z$ then $K = [K, K] + s \circ [K, K]$ where $s \circ [K, K]$ is the additive subgroup of K generated by $\{s \circ k \mid k \in [K, K]\}$.*

PROOF. We first note that $sS^2 \subset s[K, K] + S^2$ for all $s \in S$. Let $r, t, s \in S$; then

$$2str = s(tr - rt) + s(tr + rt).$$

Hence the remark follows since the characteristic $\neq 2$.

Now choose $s \in S$, $s \notin Z$. Then there exists some $r \in S$ such that $sr - rs \neq 0$; otherwise $s \in Z$ (cf. [2, Theorem 9]). Hence, let $t \in S$ so that $st - ts \neq 0$. Consider $(st - ts)S^n$ where S^n is the additive group generated by elements of form $r_1 \cdots r_n$, $r_i \in S$ for all i. It follows that

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$$\text{(I)} \qquad (st - ts)S^n \subset S^2 + s[K, K].$$

To see this, consider $(st-ts)r_1 \cdots r_n$.

$$\begin{aligned}(st - ts)r_1 \cdots r_n \\ &= s[t, r_1 \cdots r_n + r_n \cdots r_1] - t(sr_1 \cdots r_n + r_n \cdots r_1 s) \\ &\quad + (tr_n \cdots r_1 s + sr_1 \cdots r_n t) - s(tr_n \cdots r_1 + r_1 \cdots r_n t) \\ &\quad + s(r_1 \cdots r_n + r_n \cdots r_1)t.\end{aligned}$$

The latter term is in sS^2, and thus in $s[K, K]+S^2$, and the other terms on the right are in S^2, $s[K, K]$ or S. Hence we have proved (I).

Since $\bar{S}$, the subring generated by the elements of S, is A it follows that $(st-ts)A \subset S^2+s[K, K]$. Also, $[A, A] \subset S^2+s[K, K]$ and $S^2+s[K, K]=S+[K, K]+s \circ [K, K]$. Thus, for all $a, b \in A$,

$$b(st - ts)a \in S + [K, K] + s \circ [K, K],$$

since $b(st-ts)a = [b, (st-ts)a] - (st-ts)ab$. That is,

$$A(st - ts)A \subset S + [K, K] + s \circ [K, K].$$

By assumption $st-ts \neq 0$ and A is simple; therefore $A = S+[K, K] +s \circ [K, K]$ and we conclude, using the fact that $S \cap K = (0)$, that $K = [K, K]+s \circ [K, K]$ for all $s \in S$, $s \notin Z$.

We need the following lemma to prove the main theorem:

LEMMA 2. *Let U be a strong Lie ideal of K such that U contains $[K, K]$. Then if u and v are in U, $3(v^2u+uv^2)$ is in U.*

PROOF. If $u, v \in U$, then $(u+v)^3$, $(u-v)^3$ and $2u^3 \in U$. Hence, $(u+v)^3+(u-v)^3-2u^3 \in U$; that is,

$$2(v^2u + vuv + uv^2) \in U.$$

Also, $[v, [v, u]] \in U$ and hence their difference, $3(v^2u+uv^2)$, is in U.

We are now in a position to prove Theorem 1. Herstein [2] has proven that if A is as above and U is a Lie ideal of K then either $U \subset Z$ or $U \supset [K, K]$. In that which follows we assume $U \supset [K, K]$. By Remark 2 there exists $k \in [K, K] \subset U$ such that $k^2 \notin Z$ and hence, by Lemma 2,

$$3(k^2m + mk^2) \in U \text{ for all } m \in [K, K].$$

By assumption, the characteristic $\neq 3$. Thus $[K, K]=3[K, K]$ and

$$k^2n + nk^2 \in U \text{ for all } n \in [K, K].$$

However, by Lemma 1, $K = k^2 \circ [K, K]+[K, K]$ and, since $[K, K] \subset U$, it follows that $K = U$. This completes the proof of Theorem 1.

It should be noted that Lemma 1 is interesting, independent of *strong* Lie ideals, for it generalizes a result of [1]. This generalization can be stated as follows:

THEOREM 2. *Let A be a simple ring of characteristic $\neq 2$, with either its center $Z=(0)$ or of dimension greater than* 16 *over its center, and with an involution defined on it; then if either K, S, $[K, K]$ or $[K, S]$ are finite dimensional, A is finite dimensional.*

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TRIANGLE INEQUALITY IN l-GROUPS

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In [1, p. 309] G. Birkhoff remarks that "In a *commutative* l-group, we can . . . prove the triangle inequality $|a+b| \leqq |a| + |b|$, but this does not seem to hold in general." The purpose of this note is to show that in fact *if*

$$|a+b| \leqq |a| + |b| \tag{1}$$

for all a and b in the additive l-group G, then G is commutative.

PROOF. It is sufficient to show that any two positive elements of G are permutable [2, p. 234]. Suppose therefore that x and y are positive elements of G. Taking $a=-x$ and $b=-y$ in (1), we obtain $x+y \geqq |-x-y| = y+x$. Similarly $y+x \geqq x+y$. Hence $x+y=y+x$. This completes the proof.

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