INSERTION OF ± SIGNS IN $e^x$

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We consider the series $e^x = \sum x^n/|n|$ and investigate to what extent its "magnitude" can be lessened by a propitious choice of ±1 coefficients.

This investigation began with the following elementary question.  I. Can $\epsilon_n$ be chosen ($\epsilon_n = \pm 1$) such that

$$\sum \frac{\epsilon_n x^n}{n} \rightarrow 0 \text{ as } x \rightarrow -\infty \text{ and as } x \rightarrow +\infty?$$

We now prove a rather general theorem which contains the negative answer to I.

**Theorem.**

$$f(x) = \sum \frac{\epsilon_n x^n}{n} = Oe^{\rho x}, \quad \rho < 1, \text{ as } x \rightarrow +\infty$$

if and only if, for sufficiently large $n$, the $\epsilon_n$ are periodic, with even period 2$K$, and $\epsilon_{n+1} + \epsilon_{n+2} + \cdots + \epsilon_{n+2K} = 0$.

**Proof.** The "if" part is, of course, the easy half, for if the $\epsilon_n$ are periodic of period 2$K$ then $f(x)$ is a linear combination of the $e^{\omega r x}$ where $\omega = e^{\pi /K}$, $0 \leq r < 2K$. If this combination is

$$f(x) = \sum_{r=0}^{2K-1} a_r e^{\omega r x}$$

then

$$\epsilon_n = \sum_{r=0}^{2K-1} a_r \omega^{r n} \text{ and so } \epsilon_{n+1} + \cdots + \epsilon_{n+2K} = 2Ka_0$$

hence $a_0 = 0$ and so $|f(x)| \leq Me^{(\cos r /k)x}$. We now prove the "only if" statement:

Consider the expression

$$\frac{1}{Z} \int_0^\infty f(x)e^{-x/2}dx = g(Z).$$

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First note that, since \( |\gamma(x)| \leq Me^{\pi x} \), \( g(Z) \) is analytic for \( \text{Re} \left( \frac{1}{Z} \right) > \rho \). Next note that, for \( \text{Re} \left( \frac{1}{Z} \right) > 1 \),

\[
g(Z) = \frac{1}{Z} \int_0^\infty \sum \epsilon_n \frac{x^n}{n} e^{-x/Z} dx = \frac{1}{Z} \sum \epsilon_n \int_0^\infty \frac{x^n}{n} e^{-x/Z} dx,
\]

the inversion being justified by the bounded convergence theorem since

\[
\int_0^\infty \sum \left| \epsilon_n \frac{x^n}{n} \right| e^{-x/Z} dx = \int_0^\infty e^{e^{-2\text{Re} \left( \frac{1}{Z} \right)}} dx < \infty.
\]

Finally, then, for

\[
\text{Re} \left( \frac{1}{Z} \right) > 1,
\]

\[
g(Z) = \sum \frac{\epsilon_n}{Z} \int_0^\infty \frac{x^n}{n} e^{-x/Z} dx = \sum \epsilon_n Z^n.
\]

The conclusion is that \( \sum \epsilon_n Z^n \) can be continued past the unit circle [into \( \text{Re} \left( \frac{1}{Z} \right) > \rho \), in fact.]

It is a theorem of Carlson [1], however, that if the \( a_n \) are integers, \( \sum a_n Z^n \) has radius of convergence = 1, and \( \sum a_n Z^n \) is not a rational function then \( |Z| = 1 \) is the natural boundary for \( \sum a_n Z^n \).

The conclusion for us, then, is that \( \sum \epsilon_n Z^n \) is a rational function! It therefore follows that, for \( n \) past a certain point, the \( \epsilon_n \) satisfy a finite linear recurrence relation. Because of this and the fact that the \( \epsilon_n \) take on only a finite number of values (\( \epsilon_n = \pm 1 \)), it follows that the \( \epsilon_n \) are periodic past a certain point. Hence

\[
g(Z) = \sum \epsilon_n Z^n = P(Z) + \frac{\delta_0 + \delta_1 Z + \cdots + \delta_{M-1} Z^{M-1}}{1 - Z^M}
\]

where \( P \) is a polynomial, \( \delta_j = \pm 1 \), and \( M \) the period of \( \epsilon_n \). For sufficiently large \( n \) we obtain

\[
\epsilon_{n+1} + \epsilon_{n+2} + \cdots + \epsilon_{n+M} = \delta_0 + \delta_1 + \cdots + \delta_{M-1}
\]

but

\[
\delta_0 + \delta_1 + \cdots + \delta_{M-1} = \lim_{Z \to 1} (1 - Z^M) g(Z)
\]

and the latter is 0 since \( g(Z) \) is regular at 1. Hence \( \epsilon_{n+1} + \cdots + \epsilon_{n+M} = 0 \), and in particular \( M = 2K \), \( K \) integral and the proof is complete.

The negative answer to question I is now easily given. If \( f(x) \to 0 \) at \( \pm \infty \) then, since by our theorem, \( f(x) \) is a trigonometric polynomial, we have \( f(x) = Oe^{-a|x|}, \ x \to \pm \infty \). Since, however, \( |f(x)| \leq e^{c|x|} \) for all

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Certain other corollaries can be reaped. It can be shown, e.g. that if $f(x) = O(1)$, $x \to +\infty$, then $f(x)$ is equal to $\pm \sin x \pm \cos x$ or $\pm e^{-x}$.

(This result also settles question I.)

Just one final remark, and this is to state that one can estimate $K$ in terms of $\rho$ namely, $K \leq \exp \left[ C_1 (1 - \rho)^{-1/2} \right]$; this estimate is furthermore fairly good since for every $\rho < 1$ there exists $f(x)$ with $K \geq \exp \left[ C_2 (1 - \rho)^{-1/4} \right]$.

REFERENCES


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