ON THE KANTOROVICH INEQUALITY

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The version of Kantorovich's inequality given by Greub and Rheinboldt [1] is susceptible of an easy but nevertheless substantial generalization. Their result is:

If a self-adjoint operator $A$ on Hilbert space satisfies $m(x, x) \leq (Ax, x) \leq M(x, x)$ for all $x \in \mathcal{H}$, with $m > 0$, then for all $x \in \mathcal{H}$,

$$
(Ax, x)(A^{-1}x, x) \leq \frac{(M + m)^2}{4mM} (x, x)^2.
$$

Theorem. If $T$ is an arbitrary operator on $\mathcal{H}$, and $\|T\| = M$, $\|T^{-1}\| = m^{-1}$, then for all $x, y \in \mathcal{H}$,

$$
| (Tx, y)(x, T^{-1}y) | \leq \frac{(M + m)^2}{4mM} (x, x)(y, y).
$$

Furthermore, the bound is best possible.

Proof. Let $A = (T^*T)^{1/2}$, taking the unique positive definite self-adjoint square root, so that (1) holds. Then $U = TA^{-1}$ is unitary, and

$$
| (Tx, y)(x, T^{-1}y) |
= | (UAx, y)(x, A^{-1}U^{-1}y) |
= | (Ax, U^*y)(A^{-1}x, U^*y) |
\leq [(Ax, x)(AU^*y, U^*y)(A^{-1}x, x)(A^{-1}U^*y, U^*y)]^{1/2}
$$

by the generalized Schwarz inequality [2, p. 262]. Now (2) follows immediately from (1) and (3), using $(U^*y, U^*y) = (y, y)$.

If $\mathcal{H}$ is finite dimensional, the bound is attained for $x = U^*y = u + v$, where $u$ and $v$ are unit eigenvectors of $A$ corresponding to eigenvalues $m$ and $M$. In the general case, the bound need not be attained; but a sequence $x_n = U^*y_n = u_n + v_n$, where $\|u_n\| = \|v_n\|$, $(E_{m+1/n} - E_{m-0})u_n = u_n$, $(E_{M+0} - E_{M-1/n})v_n = v_n$ ($\{E\}$ being the spectral resolution for $A$) shows on calculation that the bound is best possible.

References


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468