ON THE KANTOROVICH INEQUALITY

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The version of Kantorovich's inequality given by Greub and Rheinboldt [1] is susceptible of an easy but nevertheless substantial generalization. Their result is:

If a self-adjoint operator A on Hilbert space satisfies $m(x, x) \leqq (Ax, x) \leqq M(x, x)$ for all $x \in \mathcal{H}$, with $m > 0$, then for all $x \in \mathcal{H}$,

$$(1) \qquad (Ax, x)(A^{-1}x, x) \leqq \frac{(M+m)^2}{4mM}(x, x)^2.$$

THEOREM. *If T is an arbitrary operator on $\mathcal{H}$, and $\|T\| = M$, $\|T^{-1}\| = m^{-1}$, then for all $x, y \in \mathcal{H}$*

$$(2) \qquad |(Tx, y)(x, T^{-1}y)| \leqq \frac{(M+m)^2}{4mM}(x, x)(y, y).$$

Furthermore, the bound is best possible.

PROOF. Let $A = (T^*T)^{1/2}$, taking the unique positive definite self-adjoint square root, so that (1) holds. Then $U = TA^{-1}$ is unitary, and

$$(3) \qquad \begin{aligned} &|(Tx, y)(x, T^{-1}y)| \\ &\quad = |(UAx, y)(x, A^{-1}U^{-1}y)| = |(Ax, U^*y)(A^{-1}x, U^*y)| \\ &\quad \leqq [(Ax, x)(AU^*y, U^*y)(A^{-1}x, x)(A^{-1}U^*y, U^*y)]^{1/2} \end{aligned}$$

by the generalized Schwarz inequality [2, p. 262]. Now (2) follows immediately from (1) and (3), using $(U^*y, U^*y) = (y, y)$.

If $\mathcal{H}$ is finite dimensional, the bound is attained for $x = U^*y = u + v$, where u and v are unit eigenvectors of A corresponding to eigenvalues m and M. In the general case, the bound need not be attained; but a sequence $x_n = U^*y_n = u_n + v_n$, where $\|u_n\| = \|v_n\|$, $(E_{m+1/n} - E_{m-0})u_n = u_n$, $(E_{M+0} - E_{M-1/n})v_n = v_n$ ($\{E_\lambda\}$ being the spectral resolution for A) shows on calculation that the bound is best possible.

REFERENCES

1. W. Greub and W. Rheinboldt, *On a generalization of an inequality of L. V. Kantorovich*, Proc. Amer. Math. Soc. vol. 10 (1959) pp. 407–415.

2. F. Riesz and B. Sz.-Nagy, *Functional analysis*, New York, F. Ungar Publishing Company, 1955.

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