REFERENCES

1. C. E. Capel, *Inverse limit spaces*, Duke Math. J. vol. 21 (1954) pp. 233–245.
2. S. Eilenberg and N. E. Steenrod, *Foundations of algebraic topology*, Princeton University Press, 1952.
3. Hans Freudenthal, *Entwicklungen von Raumen und ihrer Gruppen*, Compositio Math. vol. 4 (1937) pp. 145–234.
4. J. W. T. Youngs, *Homeomorphic approximations to monotone mappings*, Duke Math. J. vol. 15 (1948) pp. 87–94.

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NEW METHOD FOR EXPANSION AND CONTRACTION MAPS IN UNIFORM SPACES

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1. Introduction and definitions. In [2], Freudenthal and Hurewicz showed that if the function f, from the totally bounded metric space M onto M, has the property that $(fx, fy) \leqq (x, y)$ for each x and y in M, then f is an isometry. By amplifying the sequential argument given in [2], Rhodes (see [4]) proved that an even stronger result holds in the more general setting of uniform spaces. Using a different method, the present paper offers a theorem similar to that of Rhodes, together with a number of results concerning "expansion" maps in uniform spaces. The notation used here, which very closely approximates that of [4], has been taken from Chapter 6 of [3].

1.1. DEFINITION. If $(M, \mathfrak{U})$ is a uniform space, then a subset $\mathfrak{B}$ of $\mathfrak{U}$ will be called a basis for $(M, \mathfrak{U})$ if

(a) if $x \in M$ and $U \in \mathfrak{B}$, then $(x, x) \in U$;

(b) if $U \in \mathfrak{U}$, then U^{-1} contains a member of $\mathfrak{B}$;

(c) for each $U \in \mathfrak{U}$ there is a $V \in \mathfrak{B}$ for which $V \circ V \subset U$; and

(d) for each $U \in \mathfrak{U}$ and $V \in \mathfrak{U}$, there is a $W \in \mathfrak{B}$ for which $W \subset U \cap V$.

1.2. DEFINITION. If $\mathfrak{B}$ is a basis for the uniform space $(M, \mathfrak{U})$, then $\mathfrak{B}$ is said to be open if each of its elements is open in $M \times M$.

1.3. DEFINITION. If $\mathfrak{B}$ is a basis for the uniform space $(M, \mathfrak{U})$, then $\mathfrak{B}$ is said to be ample if, whenever $(x, y) \in U \in \mathfrak{B}$, there is a $W \in \mathfrak{B}$ for which $(x, y) \in W \subset \overline{W} \subset U$.

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1.4. DEFINITION. If $(M, \mathfrak{U})$ is a uniform space with $A \subset M$ and $U \in \mathfrak{U}$, then A is said to be a U-net if $M \subset \bigcup_{a\in A} U[a]$.

1.5. DEFINITION. A U-net is said to be minimal if no U-net has smaller cardinality.

1.6. DEFINITION. Let $\mathfrak{B}$ be a basis for the uniform space $(M, \mathfrak{U})$, and let f be a function on M into M. Then

(a) f is said to be a contraction with respect to $\mathfrak{B}$ if $(fx, fy) \in U$ whenever $(x, y) \in U \in \mathfrak{B}$;

(b) f is said to be an expansion with respect to $\mathfrak{B}$ if $(x, y) \in U$ whenever $(fx, fy) \in U \in \mathfrak{B}$;

(c) f is said to be isobasic with respect to $\mathfrak{B}$ if f is both a contraction with respect to $\mathfrak{B}$ and an expansion with respect to $\mathfrak{B}$.

1.7. REMARK. If M is a metric space with metric ρ, and if $\mathfrak{B}$ is the family of all sets of the form $U_\epsilon = \{(x, y) \mid \rho(x, y) < \epsilon\}$, with $\epsilon > 0$, then the function $f: M \to M$ is isobasic with respect to $\mathfrak{B}$ if and only if f is a ρ-isometry.

2.1. THEOREM. *Let $\mathfrak{B}$ be an open basis for the totally bounded Hausdorff uniform space $(M, \mathfrak{U})$, and let the function f, mapping M onto M, be a contraction with respect to $\mathfrak{B}$. Then $(x, y) \in \overline{U}$ whenever $(fx, fy) \in U \in \mathfrak{B}$. If $\mathfrak{B}$ is ample, then f is isobasic with respect to $\mathfrak{B}$.*

PROOF. We prove the latter of the two statements above. Given $(fx, fy) \in U \in \mathfrak{B}$, we find $W \in \mathfrak{B}$ such that $(fx, fy) \in W \subset \overline{W} \subset U$. Suppose now that $(x, y) \notin U$. Then there is (see, for example, Theorem 7, p. 179 of [3]), a symmetric $V_1 \in \mathfrak{U}$ for which $(x, y) \notin V_1 \circ W \circ V_1$. Choosing $V_2 \in \mathfrak{U}$ so that $V_2[fx] \times V_2[fy] \subset W$ and selecting $V \in \mathfrak{B}$ so that $V \cup V^{-1} \subset V_1 \cap V_2$, we see that if $(x, x') \in V$ and $(y, y') \in V$, then $(x', y') \notin W$ and $(fx', fy') \in W$.

Now for each minimal V-net $A = \{a_1, \cdots, a_n\}$ in M, write $(i, j) \in D(A)$ whenever $1 \leqq i < j \leqq n$ and $a_j \in W[a_i]$. Let $R(A) = \operatorname{card} D(A)$, and choose A to be a minimal V-net for which $R(A)$ is maximal. Since there is no index i for which $x \in V[a_i]$ and $y \in V[a_i]$, we may suppose the notation chosen so that $x \in V[a_1]$ and $y \in V[a_2]$. Then $(1, 2) \notin D(A)$. Now for $1 \leqq i \leqq n$ let $b_i = fa_i$, and let $B = \{b_1, \cdots, b_n\}$. Then B is a minimal V-net, and $(1, 2) \in D(B)$. But $(i, j) \in D(B)$ whenever $(i, j) \in D(A)$, so that $R(B) > R(A)$. This contradiction completes the proof.

2.2. COROLLARY. EDREI (*See Theorem* 1 *of* [1]). *Let M be a totally bounded metric space and let the function f map M onto M. Suppose that there exists $\epsilon > 0$ for which $(fx, fy) < \epsilon$ whenever $(x, y) < \epsilon$. Then $(fx, fy) = (x, y)$ whenever $(x, y) < \epsilon$.*

3. **Expansion maps in uniform spaces.** In this section we show, by employing a device similar to that of Theorem 2.1, that an expansion map on a compact Hausdorff space into itself must be an onto mapping.

3.1. PROPOSITION. *Let the uniform space* $(M, \mathfrak{U})$ *be compact Hausdorff, and let* $\mathfrak{B}$ *be a basis for* $(M, \mathfrak{U})$. *Let* $f: M \to M$ *be an expansion with respect to* $\mathfrak{B}$, *and suppose that* $f(x_\alpha)$ *and* $f(y_\beta)$ *are nets in* fM *which converge to a common point. Then the nets* x_α, y_β *converge to a common point.*

3.2. LEMMA. *Let the uniform space* $(M, \mathfrak{U})$ *be totally bounded Hausdorff, and let* $\mathfrak{B}$ *be a basis for* $(M, \mathfrak{U})$. *Let* $f: M \to M$ *be an expansion with respect to* $\mathfrak{B}$. *Then* fM *is dense in* M.

PROOF. Suppose that there exist $p \in M$ and $U \in \mathfrak{B}$ for which $U[p] \subset M \setminus fM$. For each subset F of M, write $F \in \mathfrak{F}$ if $(x, y) \in U$ whenever $x \in F$ and $y \in F$. The family $\mathfrak{F}$ admits a finite subcover of M, since if $V \in \mathfrak{B}$ and $V \circ V \subset U$, then $\{\text{int } V[x] \mid x \in M\}$ covers M. Let $\{F_1, \cdots, F_k\}$ be a subset of $\mathfrak{F}$ which covers M, selected so that k is minimal. We may suppose that $p \in F_k$. Then $F_k \subset U[p]$, so that $f^{-1}F_k = \Lambda$. If $1 \leqq i \leqq k-1$, then $f^{-1}F_i \in \mathfrak{F}$; and $\bigcup_{i=1}^{k-1} f^{-1}F_i = M$, contrary to the minimality of k.

3.3. DEFINITION. Let the uniform space $(M, \mathfrak{U})$ be compact Hausdorff, and let $\mathfrak{B}$ be a basis for $(M, \mathfrak{U})$. Let $f: M \to M$ be an expansion with respect to $\mathfrak{B}$, and suppose that $f(x_\alpha)$ is a net in fM which approaches the point $p \in M$. Then the limit in M of the net x_α is called the pseudo-inverse (under f) of p, and is denoted p^*.

3.4. PROPOSITION. *Let* M, $\mathfrak{U}$, $\mathfrak{B}$ *and* f *be as in* 3.3, *and suppose that* $\mathfrak{B}$ *is ample. Then* $(x^*, y^*) \in U$ *whenever* $(x, y) \in U \in \mathfrak{B}$.

3.5. THEOREM. *Let the uniform space* $(M, \mathfrak{U})$ *be compact Hausdorff, and let* $\mathfrak{B}$ *be an ample basis for* $(M, \mathfrak{U})$. *Let* $f: M \to M$ *be an expansion with respect to* $\mathfrak{B}$. *Then* $f(p^*) = p$ *for each* $p \in M$, *so that* $fM = M$.

PROOF. If there is a $p \in M$ for which $f(p^*) \neq p$, then there is a $U \in \mathfrak{B}$ for which $(f(p^*), p) \notin U$. We select $V \in \mathfrak{B}$ so that $V \circ V^{-1} \circ V \circ V^{-1} \subset U$.

Now for each minimal V-net $A = \{a_1, \cdots, a_n\}$ in M, write $(i, j) \in E(A)$ whenever $1 \leqq i < j \leqq n$ and $V[a_i] \cap V[a_j] \neq \Lambda$. Let $S(A) = \text{card } E(A)$, and choose A to be a minimal V-net for which $S(A)$ is maximal. Since there is no index i for which $p \in V[a_i]$ and $f(p^*) \in V[a_i]$, we may suppose the notation chosen so that $p \in V[a_1]$ and

$f(p^*) \in V[a_2]$. Then $(1, 2) \notin E(A)$. Let $B = \{a_1^*, \cdots, a_n^*\}$. Since $x = (fx)^*$ for each $x \in M$, it follows from 3.4 that B is a minimal V-net in M and that $p \in V[a_1^*] \cap V[a_2^*]$. Hence $(1, 2) \in E(B)$. But $(i, j) \in E(B)$ whenever $(i, j) \in E(A)$, so that $S(B) > S(A)$. This contradiction completes the proof.

3.6. COROLLARY. BANACH-ULAM (*See Theorem* 3 *of* [**5**]). *A compact metric space cannot be isometric with a proper subset of itself.*

3.7. COROLLARY. *Let the uniform space* $(M, \mathcal{U})$ *be compact Hausdorff, and let* $\mathcal{B}$ *be an ample, open basis for* $(M, \mathcal{U})$. *Let* $f: M \rightarrow M$ *be an expansion with respect to* $\mathcal{B}$. *Then* f *is isobasic with respect to* $\mathcal{B}$.

PROOF. The function f is clearly (1-1), and its inverse function f^{-1} exists by 3.5. Since f^{-1} satisfies the hypotheses of 2.1, f^{-1} is isobasic with respect to $\mathcal{B}$. Hence f is also.

3.8. COROLLARY. *Let* M *be a compact metric space and let* $f: M \rightarrow M$. *Suppose that there exists* $\epsilon > 0$ *for which* $(fx, fy) \geqq (x, y)$ *whenever* $(x, y) < \epsilon$. *Then* $(fx, fy) = (x, y)$ *whenever* $(x, y) < \epsilon$.

REFERENCES

1. Albert Edrei, *On mappings which do not increase small distances*, Proc. London Math. Soc. (3) vol. 2 (1952) pp. 272–278.

2. H. Freudenthal and W. Hurewicz, *Dehnungen, Verkurzungen, Isometrien*, Fund. Math. vol. 26 (1936) pp. 120–122.

3. John L. Kelley, *General topology*, New York, Van Nostrand, 1955.

4. F. Rhodes, *A generalization of isometries to uniform spaces*, Proc. Cambridge Philos. Soc. vol. 52 (1956) pp. 399–405.

5. S. Ulam, *The Scottish book of Lwów*, mimeographed.

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