DIFFERENTIABLY SIMPLE RINGS

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Let $R$ be a ring and $\mathcal{D}$ a family of derivations of $R$ into itself. We call $R$ differentiably simple under $\mathcal{D}$ if $R^2 \neq 0$ and if $R$ has no two-sided ideal (other than 0 and $R$) sent into itself under every derivation of the family $\mathcal{D}$ (i.e., has no differential ideal). We shall call $R$ differentiably simple, usually without specifying $\mathcal{D}$. The purpose of this paper is to explore the analogy between simple rings and differentiably simple rings.

1. Structure theory.

Theorem 1. Let $R$ be differentiably simple; then $R^2 = R$; also, $R$ has no absolute left or right divisors of zero.

Proof. $R^2$ is a nonzero differential ideal, for every derivation. To prove the second part, let for example, $aR = 0$, then $d(aR) = 0 = d(a)R + ad(R) = d(a)R$ so $d(a)R = 0$ so that the absolute left zero divisors are a (two-sided) differential ideal of $R$; since $R^2 \neq 0$, this ideal is zero.

Theorem 2. Let $R$ be differentiably simple and let $F$ be the set of those elements of the centroid of $R$ commuting with every derivation of $\mathcal{D}$; then $F$ is a field, called the differential centroid of $R$; if $1 \in R$, $F$ is the subset of $R$ annihilated by every element of $\mathcal{D}$.

Proof. $F$ is contained in the centroid, which is commutative since $R^2 = R$; if $b$ is a nonzero element of $F$, $b(R)$ is a nonzero differential ideal of $R$, etc.

Lemma. A differentiably simple ring $R$ is not locally nilpotent.

Proof. Consider for any $a$ not zero in $R$ the nonzero differential ideal consisting of all sums of two-sided multiples of all products of two derivatives of $a$ of any orders and mixture. This ideal is then the entire ring, so $a$ is in it. Let $n$ be the largest number of derivations occurring in one of the above-mentioned derivatives. We may suppose $n > 0$. Consider the set of derivatives $d_1, \ldots, d_k$ which occur at all in the expression for $a$; differentiate this expression by every product $d_{i_1}d_{i_2}\cdots d_{i_l}$, $0 \leq l \leq 2n - 1$, $1 \leq i_j \leq k$, $1 \leq j \leq l$. Then $d_{i_1}d_{i_2}\cdots d_{i_l}(a) = \sum r_{i_1}d_{i_1}\cdots d_{i_l}(a)s_{i_1}\cdots s_{i_l}$ for every $(i_1, \ldots, i_l)$, $0 \leq l \leq 2n - 1$ for every $1 \leq i_j \leq k$, the sum being extended over every
$q$-tuple $(j_1 \cdots j_q)$, $1 \leq j_q \leq k$, $0 \leq q \leq 2n - 1$. Since the subring of $R$ generated by the $r_{j_1\cdots j_q}$ and $s_{j_1\cdots j_q}$ is nilpotent, by repeated use of this set of equations we find $a = 0$ in contradiction to hypothesis.

**Definitions.** An ideal divisor of zero in a ring is a two-sided ideal annihilated by a nonzero element, on the right or on the left. A ring is called primary if every ideal divisor of zero is nilpotent.

**Theorem 3.** A differentiably simple ring is primary.

**Proof.** Suppose $I$ is a two-sided ideal of $R$ and $Ib = 0$, $b \neq 0$. We shall show that the Levitzki Nil Radical of $R$ contains a differential ideal if $I^n = 0$ for no $n$. $a \in I \Rightarrow axb = 0 \forall x \in R$, so $d(a_1)xb + a_1xd(b) = 0 \forall x \in R$, $a_2d(a_1)x + a_3dx(b) = 0$. In general, $a_1a_2a_3 \cdots d(a_1)x + a_2a_3a_4 \cdots d(a_2)x + \cdots + a_{n-1}a_n = 0 \forall a_1, \cdots, a_n \in I, d_1, \cdots, d_n \in D$. For $a \neq 0$, let $J = \{q/axq = 0 | x \in R\}$. $J$ is a two-sided ideal of $I$; we shall show that $J$ is locally nilpotent. For let $K = \{c \in R | cx - 1x7 - 1 < \sigma = 0, \forall x \in R, \text{for all products of any length} m = m(c) \text{chosen from a fixed finite set} q_1, \cdots, q_r\}$. $K$ is a two-sided ideal of $R$, in fact a differential one (where $m(d(c)) = 2m(c)$), and $K \neq 0$, so $K = R$. A fortiori, every product of length $t$ of the $q_i$ is zero, where $t = 2 + \max_{1 \leq i \leq r} m(q_i)$. That is, $J$ is locally nilpotent.

Since for any $n$ there is a product of $a_1, a_2, \cdots, a_n$ which is not zero, every mixed derivative of $b$ is in the Levitzki Nil Radical of $R$ (the maximal locally nilpotent ideal of $R$). $R$ itself would then be locally nilpotent, which contradicts the lemma. So $I^n = 0$ for some $n$, which proves the theorem.

**Theorem 4.** Let $R$ be a differentiably simple ring whose differential centroid is of characteristic zero. Then $R$ is a prime ring, i.e., there are no ideal divisors of zero.

**Proof.** We prove more generally that if $R$ is a primary ring whose additive group is torsion-free, and $a$ lies in a nilpotent ideal, then $d(a)$ lies in a nilpotent ideal, for every derivation $d$ of $R$. So let $n$ be the smallest integer such that $ax_1ax_2 \cdots ax_n = 0 \forall x_i \in R, 1 \leq i \leq n$. Assume $d(a)$ does not lie in a nilpotent ideal. Differentiate the preceding equation $n$ times, and left multiply by $a_1a_2a_3 \cdots a_{n-1}$, where $a_i$, $1 \leq j \leq n - 1$ are any elements of $R$. We obtain $(n + 1)a_1a_2 \cdots a_{n-1}d(a)x_id(a)x_2 \cdots d(a)x_n = 0, a_1a_2a_3 \cdots a_{n-1}d(a)x_1d(a)x_2 \cdots d(a)x_n = 0 \forall x \in R, 1 \leq j \leq n - 1, \forall x \in R, 1 \leq i \leq n$. For simplicity, right multiply by $d(a)$; then $d(a)$ lies in an ideal divisor of zero and hence in a nilpotent ideal, or else $a_1a_2 \cdots a_{n-1}d(a)x_1 \cdots x_n = 0$. This latter case must occur. Continuing this stripping off process, we find $a_1a_2 \cdots a_{n-1}d(a) = 0$, $\forall x_1, \cdots, x_{n-1} \in R$. Then $d(a)$ lies in
an ideal divisor of zero, which is impossible, or else \( at_1 at_2 \cdots at_{n-1} = 0 \forall t_1 \cdots t_{n-1} \in R \), which contradicts the minimality of \( n \).

Then every differentiably simple ring of characteristic zero is prime. For every ideal divisor of zero is nilpotent, so that if \( R \) has no nilpotent ideals, we are done; if \( R \) has, however, a nilpotent ideal, we have just shown that the sum of the nilpotent ideals of \( R \) is a differential ideal, hence all of \( R \). Then \( R \) would be locally nilpotent, which we know to be impossible.

For an application of this result to algebraic functions see [1].

**Corollary.** A differentiably simple ring of characteristic zero with a minimal two-sided ideal is simple. In particular if \( R \) satisfies the descending chain condition for left or right ideals, \( R \) is simple.

**Proof.** Let \( R \) be differentiably simple, and \( I \) a minimal 2-sided ideal \( \neq 0 \). \( I^2 \neq 0 \) since \( R \) is prime, so \( I^2 = I \). Then \( I \) is differential (for any derivation), so \( I = R \), \( R \) is simple.

**Theorem 5.** Every commutative differentiably simple ring has a unit.

**Proof.** We must separate the characteristic \( p \) and zero cases. First let \( R \) be differentiably simple under \( \mathcal{D} \) and of characteristic \( p \neq 0 \). For every \( x \in R \), \( d \in \mathcal{D} \), \( d(x^p) = 0 \), so (here is where we use commutativity) \( x^p \) is in the differential centroid of \( R \). If every \( x^p = 0 \), then \( R \) is a commutative nil ring, hence locally nilpotent. But a differentiably simple ring is not locally nilpotent. Hence for at least one \( x \in R \), \( x^p \neq 0 \) and \( R \) contains a nonzero element of the differential centroid, and hence a unit.

Now let \( R \) be commutative differentiably simple and of characteristic zero. We shall assume that \( \mathcal{D} = \{ d \} \), but only to simplify notation. Look at the proof of the lemma in Theorem 3. Let \( a \) be any nonzero element of \( R \). Then there are elements \( r_{ij} \in R \) such that

\[
a = \sum_{i=0}^{n} \sum_{j=0}^{n} r_{ij} a^{(i)} a^{(j)}.
\]

Differentiate \( 2n - 1 \) times (without loss of generality, we can assume \( n > 0 \)). Then

\[
a^{(k)} = \sum_{t=0}^{2n-2} l_{k,t} a^{(t)}, \quad 0 \leq k \leq 2n-1, t_{k,t} \in R.
\]

Consider the column vector with \( 2n \) components, \( v \), whose \( j \)th component is \( a^{(2n-1)} \), and the matrix \( T = (t_{k,t}) \). Then \( v = Tv \). Regard \( T \) as a matrix over the quotient field of \( R \) (\( R \) has no divisors of zero since it is a commutative prime ring). \( (T-I)v = 0 \).

But \( v \neq 0 \) since \( a \neq 0 \), \( T-I \) is singular.

So \( \det(T-I) = 0 \). Expand this out. Every term except the product of the 1's on the diagonal contains at least one element of \( R \) and the rest 1's. Then \( r - (-1)^{2n} e_{R} \) where \( r \in R \) so the unit 1 of the quotient field actually appears as an element of \( R \), \( R \) has a unit.
Extensions of differentiably simple rings.

Theorem 6. Let $R$ be a commutative ring differentiably simple under $\mathcal{D}$ with differential centroid $F$. Let $M$ be a nonempty multiplicatively closed subset of $R$ containing 1 but not zero. Let $R_M$ be the ring of quotients of $R$ with respect to $M$. Let $\mathcal{D}_M$ be the set of derivations of $\mathcal{D}$ uniquely extended to $R_M$. Then $R_M$ is differentiably simple under $\mathcal{D}_M$ with differential centroid $F$.

Proof. Let $I$ be a nonzero differential ideal of $R_M$; then the set of $a$ in $R$ for which $b \in M$ with $a/b \in I$ is a nonzero differential ideal of $R$ contained in $I$. So $R \subseteq I$, $I = R_M$. To prove $R_M$ has differential centroid $F$, we invoke the following lemma.

Lemma. A commutative differentiably simple ring $R$ is differentially closed in its full ring of quotients $S$.

Proof. We are to prove that if $a \in S$ and $d(a) \in R \forall d \in \mathcal{D}$, then $a \in R$. Let $J = \{x \in R/xa \in R\}$. $J$ is a differential ideal of $R$ and is not zero. Then $J = R$, $1 \cdot a \in R$, $a \in R$.

Returning to the theorem, we are to prove that if $d(a) = 0 \forall d \in \mathcal{D}$, then $a \in F$. But by the Lemma, $a \in R$, whence $a \in F$.

As an application of this theorem, consider a differential field $L$. (The differential centroid of a differential field is also called its field of constants.) We wish to prove the known result from differential algebra that an integral can be adjoined to a differential field with the addition of no new constants. Consider $L$ a differential field, $x$ a differential indeterminate over $L$, and, in the field $L(x)$, extend the original derivation by $x' = a$, where $a$ was not a derivative in $L$. We wish to show that $L(x)$ has no new constants. However, it can be shown that $L[x]$ is differentiably simple and has the same differential centroid as $L$. The quotient field $L(x)$ of $L[x]$ has the same differential centroid as $L[x]$, and hence the same as $L$. That is, $L(x)$ has no new constants.

Theorem 7. Let $R$ be differentiably simple under $\mathcal{D}$ with differential centroid $F$. Let $K$ be a field containing $F$. Then $R \otimes_F K$ (under the natural extension of $\mathcal{D}$) is differentiably simple, with differential centroid $K$.

Proof. Consider the subring of the full ring of $F$-endomorphisms of $R$ generated by left and right multiplications by elements of $R$ and by differentiation by elements of $\mathcal{D}$. To say that $R$ is differentiably simple under $\mathcal{D}$ is the same as saying that $R$ is irreducible under this ring. Furthermore, the differential centroid is just the commuting ring of this ring of endomorphisms. The tensor product...
over $F$ of this ring and $K$ is the analogous ring for $R \otimes_F K$. Reference [2] is just what we need to say that $R \otimes_F K$ is irreducible under this tensored ring with commuting ring $K$, i.e., differentiably simple with differential centroid $K$.

3. The ring of differential polynomials. We are going to study the above ring of endomorphisms when $R$ is commutative. First consider the ring of differential polynomials over $\mathcal{D}$ with coefficients in $R$, $A_\mathcal{D}(R)$. This will be the ring of polynomials in noncommuting indeterminates, one for each derivation in $\mathcal{D}$, with coefficients from $R$ written on the left. Here multiplication is defined by $Da = d(a) + aD$ where $D$ is the indeterminate corresponding to the derivation $d \in \mathcal{D}$. Then $A_\mathcal{D}(R)$ is an (associative) ring, and what is more, $R$ is a left $A_\mathcal{D}(R)$ module by $D(a) = d(a)$. $R$ is contained in $A_\mathcal{D}(R)$ as polynomials of degree zero. Each $d \in \mathcal{D}$ extends to the inner derivation of $A_\mathcal{D}(R)$ given by $x \rightarrow Dx - xD$. $R$ is an irreducible left $A_\mathcal{D}(R)$ module if and only if $R$ is differentiably simple under $\mathcal{D}$. Let $B_\mathcal{D}(R)$ be $A_\mathcal{D}(R)$ made faithful on $R$. (This is the ring of the preceding section.) $R$ is still contained in $B_\mathcal{D}(R)$ since $R$ has no absolute left divisors of zero. $F$ is the commuting ring of endomorphisms of $B_\mathcal{D}(R)$ acting on $R$, and also of $A_\mathcal{D}(R)$ acting on $R$. Let $N$ be the ideal of $A_\mathcal{D}(R)$ annihilating $R$, so that $A_\mathcal{D}(R)/N = B_\mathcal{D}(R)$. $B_\mathcal{D}(R)$ is left primitive, $R$ being a faithful irreducible left module.

**Theorem 8.** Let $R$ be a commutative differentiably simple ring of characteristic $p$ ($\neq 0$ forced) of finite dimension over its differential centroid $F$. Then $B_\mathcal{D}(R)$ is the full ring of $F$-linear transformations of $R$ into itself. If $R \not\cong F$, $[R : F] = O(p)$; if $\mathcal{D}$ contains but one derivation, $[R : F]$ is a power of $p$.

**Proof.** $B_\mathcal{D}(R)$ is the full ring of $F$-linear transformations of $R$ into itself, by the density theorem. Let $R \not\cong F$, so that $\mathcal{D}$ contains a nonzero derivation. Consider $\{d(a), d \in \mathcal{D}, a \in R\}$: there is a non-nilpotent element in this set, since $R$ is not nil. Let $d(a)$ be not nilpotent. Let $D$ be the indeterminate corresponding to $d$. Then $(Da - aD)^p = (d(a))^p = \lambda \in F$, and $\lambda$ is a nonzero commutator in $B_\mathcal{D}(R)$, namely $D(a(Da)^{p-1}) - (a(Da)^{p-1})D$. The trace of $\lambda$ is zero, $[R : F] = O(p)$. If $\mathcal{D} = \{d\}$, let $k$ be the least degree of elements of $N$. Let $J(\subseteq R)$ be the set consisting of zero and coefficients of $D^k$ for some element of degree $k$ in $N$. $J$ is an ideal of $R$, and if $a_0 + a_1 D + \cdots + a_k D^k \in N$, $D(a_0 + a_1 D + \cdots + a_k D^k) - (a_0 + a_1 D + \cdots + a_k D^k)D \in N$ and equals $\cdots + (\cdots )D^{k-1} + d(a_k)D^k$, so that $J$ is a nonzero differential ideal of $R$. Hence $J$ contains an element $b_0 + b_1 D + \cdots + b_{k-1}$
$+ D^k$. Let $m = [R : F] > 1$. dim$_F A_{D}(R)/N = m^2$ by the first part of the theorem, but now dim$_F A_{D}(R)/N \leq km$ so $m \leq k$. Since $D$ has a characteristic polynomial of degree $\leq m$, $k \leq m$, $k = m$. (Then $b_0, b_1, \ldots, b_{k-1} \in F$ and $X = b_0 + b_1 D + b_{m-1} D^{m-1} + D^m$ is the characteristic and minimal polynomial for $D$.) For all $a \in R$, $Xa - aX \in N$ and is of degree less than $m$, hence is zero.

$$Xa - aX = (X - b_0)(a) + \cdots + \left[ b_{m-1} C_{m-1,1} d(a) + C_{m,2} d^2(a) \right] D^{m-2} + C_{m,1} d(a) D^{m-1}.$$ 

Then

$$C_{m,1} D = 0, \ C_{m,1} = O(\ell); \quad b_{m-1} C_{m-1,1} D + C_{m,2} D^2 = 0,$$

in particular,

$$C_{m,2} = O(\ell), \ \ldots, \ C_{m,m-1} = O(\ell).$$

So all nontrivial binomial coefficients of $m$ are multiples of $\ell$, $m$ is a power of $\ell$, as promised.

We have seen that $B_D(R)$ is simple if $R$ is finite dimensional over its differential centroid; it is always simple.

**Theorem 9.** If $R$ is a commutative ring differentially simple under $D$ with differential centroid $F$, then $B_D(R)$ is simple with centroid $F$.

**Proof.** Obviously $(B_D(R))^2 \neq 0$. We must show that if $I$ is a two-sided ideal of $A_D(R)$, then either $I$ is contained in $N$, or $I + N = A_D(R)$. Let $x$ be a nonzero element of $I$ whose degree is minimal with respect to having a nonzero constant term $a$ (if any such exist at all). Then for all $b \in R$, $xb - bx \in I$, is of degree less than $x$, and has as constant term $(x-a)(b)$. Then $(x-a)(b) = 0$ for all $b \in R$, $x-a \in N$, $I + N$ contains the nonzero element $a$ of $R$. For all $d \in \mathfrak{D}$, $Da - aD \in I + N$, $d(a) \in I + N$, $(I + N) \cap R$ is a nonzero differential ideal and contains $1$, $I + N = A_D(R)$. If on the other hand every element of $I$ has zero constant term, then for all $b \in R$, $xb - bx \in I$ with constant term $x(b)$. Then $x(b) = 0, x \in N$ if $x \in I, I \subset N$.

The above paragraph proves that no element of $N$ has nonzero constant term, for otherwise $A_D(R) = N$. Let $a, b$ be in the centroid of $B_D(R)$ and $a, b$ be in $R$. $\alpha(ab) = \alpha(a)b, \alpha(ba) = bx(a)$, so $\alpha(a)$ commutes with elements of $R$. If $y$ in $A_D(R)$ is $\alpha(a)(N)$, then $yb - by \subseteq \mathfrak{N} \subseteq R$. But $yb - by$ has constant term $(y-y_0)(b)$, where $y_0$ is the constant term of $y$. So $(y-y_0)(b) = 0 \forall b \in R, y - y_0 \in N, \alpha(a) \in R$ in $B_D(R)$ (being $\equiv (\mod N)$ to $y_0$). That is, by abuse of language, $\alpha$ is in $R$, since $R$ has a unit. The proof that $d(\alpha) = 0 \forall d \in \mathfrak{D}$ is straightforward and omitted.
Remark. If $F$ is a commutative differential ring with no absolute divisors of zero such that $B_{\Delta}(R)$ is simple, $R$ is differentiably simple. The proof is similar.

**Theorem 10.** Let $R$ be a commutative differentiably simple ring of characteristic zero such that the elements of $\Delta$ commute with one another. Let $C_{\Delta}(R)$ be the ring of differential polynomials over $R$ in commuting indeterminates. Then $C_{\Delta}(R)$ is simple if and only if the elements of $\Delta$ are left linearly independent over $R$.

The proof is left to the reader.

**Bibliography**


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