NOTE ON NÖRLUND’S POLYNOMIAL $B_n^{(q)}$

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1. Nörlund [2, p. 146] has defined the polynomial $B_n^{(q)}$ by means of

$$\left( \frac{x}{e^x - 1} \right)^q = \sum_{n=0}^\infty B_n^{(q)} \frac{x^n}{n!}.$$  

Thus $B_n^{(q)}$ is a polynomial in $x$ of degree $n$ with rational coefficients; it should not be confused with the Bernoulli polynomial $B_n(x)$ defined by

$$\frac{x e^x - xe^x}{e^x - 1} = \sum_{n=0}^\infty B_n(x) \frac{x^n}{n!}.$$  

The Stirling numbers $S_1(n, k)$ and $S_2(n, k)$ of the first and second kind, respectively, are related to Nörlund’s polynomial by means of

$$(-1)^k S_1(n - 1, k) = \binom{n - 1}{k} B_k^{(n)},$$

$$S_2(n, k) = \binom{n + k}{k} B_k^{(-n)},$$

where, to begin with, $n$ is a positive integer in (2) and (3). The formulas, however, may be used to define $S_1(n, k), S_2(n, k)$ for arbitrary $n$; $k$ is restricted to integral values $\geq 0$. In particular (2) and (3) imply the reciprocity relations

$$S_1(-n - 1, k) = S_2(n, k), \quad S_2(-n - 1, k) = S_1(n, k).$$

Gould [1] has proved the elegant formula

$$B_k^{(s)} = \sum_{j=0}^k (-1)^j \binom{k + 1}{j + 1} B_k^{(-j s)},$$

which, in view of (2) and (3), yields

$$(-1)^k S_1(n - 1, k)$$

$$= \binom{n - 1}{k} \sum_{j=0}^k (-1)^j \binom{k + 1}{j + 1} \binom{k + j n}{k}^{-1} S_2(j n, k),$$

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\((-1)^k S_2(n, k)\)

\[= \binom{k + n}{k} \sum_{j=0}^{k} (-1)^j \binom{k + 1}{j + 1} \binom{j n - 1}{k}^{-1} S_1(j n - 1, k).\]

He has also proved that

\[(-1)^k \binom{z}{k} B_k^{(z-z)} = \sum_{j=0}^{k} \binom{k - z}{k + j} \binom{k + z}{k - j} \binom{k + j - 1}{k} B_k^{(i+j-k)},\]

which yields

\[S_1(n - 1, k) = \sum_{j=0}^{k} \binom{k - n}{k + j} \binom{k + n}{k - j} S_2(j, k),\]

\[S_2(n - k, k) = \sum_{j=0}^{k} \binom{k - n}{k + j} \binom{k + n}{k - j} S_1(k + j - 1, k).\]

Of these (9) is due to Schläfli, while (10) is presumably new.

2. It may be of interest to point out that (5) can be proved rapidly as follows. Since, as observed above, $B_n^{(z)}$ is a polynomial in $z$ of degree $n$, it follows from a familiar formula in finite differences that

\[\sum_{s=0}^{k+1} (-1)^s \binom{k + 1}{s} B_k^{(z-s)} = 0\]

for all $x, z$. If we take $x = z$, this becomes

\[B_k^{(z)} - \sum_{j=0}^{k} (-1)^j \binom{k + 1}{j + 1} B_k^{(-jz)} = 0,\]

which is the same as (5).

As for (8), if we put

\[g(z) = (-1)^k \binom{z}{k} B_k^{(z-z)},\]

then $g(z)$ is a polynomial in $z$ of degree $2k$. Consequently it will suffice to show that (8) holds for $2k + 1$ distinct values of $z$. For $z=0, 1, \ldots, k-1$, it is evident that $g(z) = 0$; since also

\[\binom{k - z}{k + j} \binom{k + j - 1}{k} = 0\]

\[0 \leq z < k; 0 \leq j \leq k,\]

it follows that (8) holds for these values of $z$. For $z = k$, we get
\[ (-1)^k B_{\lambda}^{(0)} = \sum_{j=0}^{k} \binom{k}{k+j} \binom{k}{k-j} \binom{k+j-1}{k} B_{\lambda}^{(j+k)}, \]

which is correct in view of
\[ \text{(11)} \quad B_{0}^{(0)} = 1, \quad B_{k}^{(0)} = 0 \quad (k \geq 1). \]

Finally for \( z = -s \), where \( s = 1, 2, \ldots, k \) we remark that the right member of (8) reduces to a single term, namely
\[ \binom{k+s}{k} \binom{k-s}{k} \binom{k+s-1}{k} B_{\lambda}^{(s+k)} = (-1)^k \binom{-s}{k} B_{\lambda}^{(s+k)}, \]

so that (8) holds in this case also. We have therefore verified that (8) is satisfied for the \( 2k+1 \) values 0, ±1, · · · , ±\( k \).

3. Examination of the above proofs reveals the somewhat surprising fact that the only property of \( B_{\lambda}^{(0)} \) that we have made use of is that \( B_{\lambda}^{(0)} \) is a polynomial in \( z \) of degree \( k \) which satisfies (11). We have therefore the following generalization. Let \( f_{k}(z) \) denote an arbitrary polynomial in \( z \) of degree \( k \). Then it follows that
\[ f_{k}(z) = \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} f_{k}(-jz). \]

If moreover
\[ \text{(13)} \quad f_{k}(0) = 0 \quad (k \geq 1), \]

then we have also
\[ (-1)^k \binom{z}{k} f_{k}(k - z) \]
\[ \text{(14)} \quad = \sum_{j=0}^{k} \binom{k-z}{k+j} \binom{k+z}{k-j} \binom{k+j-1}{k} f_{k}(j+k). \]

In addition if we define
\[ \text{(15)} \quad (-1)^k F_{1}(n - 1, k) = \binom{n-1}{k} f_{k}(n), \]
\[ \text{(16)} \quad F_{2}(n, k) = \binom{n+k}{k} f_{k}(-n), \]

then (12) and (14) yield
\[ (-1)^k F_1(n - 1, k) = \binom{n - 1}{k} \sum_{j=0}^{k} (-1)^i \binom{k + 1}{j + 1} \binom{k + jn}{k}^{-1} F_2(j n, k), \]

\[ (-1)^k F_2(n, k) = \binom{k + n}{k} \sum_{i=0}^{k} (-1)^i \binom{j + 1}{j + 1} \binom{jn - 1}{k}^{-1} F_1(j n - 1, k), \]

\[ F_1(n - 1, k) = \sum_{j=0}^{k} \binom{k - n}{k + j} \binom{k + n}{k - j} F_2(j, k), \]

\[ F_2(n - k, k) = \sum_{j=0}^{k} \binom{k - n}{k + j} \binom{k + n}{k - j} F_1(k + j - 1, k). \]

Note also that (15) and (16) imply

\[ F_1(-n - 1, k) = F_2(n, k), \quad F_2(-n - 1, k) = F_1(n, k). \]

**References**


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