ON LIE ALGEBRAS OF CLASSICAL TYPE

RICHARD BLOCK

G. B. Seligman has proved in [1] the following result: If \( \mathfrak{g} \) is a simple restricted Lie algebra over an algebraically closed field of characteristic \( p > 7 \), and if \( \mathfrak{g} \) has a restricted representation with non-degenerate trace form, then \( \mathfrak{g} \) is of classical type. By an algebra of classical type is meant an analogue over a field of characteristic \( p \) of one of the simple Lie algebras (including the five exceptional algebras) of characteristic 0; for the precise statement, see [1]. We shall show here that the above result of Seligman may be proved without the assumption of restrictedness of the algebra and its representation.

We begin with a lemma on matrices. Let \( \mathcal{M}_n(\mathbb{F}) \) denote the space of all \( n \times n \) matrices over a field \( \mathbb{F} \), and consider this as a Lie algebra (under commutation) over \( \mathbb{F} \).

**Lemma.** Let \( \mathfrak{g} \) be a (Lie) subalgebra of \( \mathcal{M}_n(\mathbb{F}) \), such that the trace form \( f(A, B) = \text{tr}(AB) \) is nondegenerate on \( \mathfrak{g} \), and let \( \mathfrak{R} \) be the normalizer in \( \mathcal{M}_n(\mathbb{F}) \) of \( \mathfrak{g} \). Then \( \mathfrak{g} \) is a direct summand of \( \mathfrak{R} \).

**Proof.** Let \( \mathcal{Q} \) be the set of all \( D \) in \( \mathfrak{R} \) such that \( f(A, D) = 0 \) for all \( A \) in \( \mathfrak{g} \). Every \( C \) in \( \mathfrak{R} \) defines a linear functional \( A \to f(A, C) \) on \( \mathfrak{g} \), so by the nondegeneracy of \( f \) on \( \mathfrak{g} \), there is a \( B = B(C) \) in \( \mathfrak{g} \) such that \( f(A, B) = f(A, C) \) for all \( A \) in \( \mathfrak{g} \). But then \( C - B \in \mathcal{Q} \) and \( C = B + (C-B) \), so \( \mathfrak{R} = \mathfrak{g} + \mathcal{Q} \). Furthermore this sum is a vector space direct sum, since \( \mathfrak{g} \cap \mathcal{Q} = 0 \) by the nondegeneracy of \( f \) on \( \mathfrak{g} \). But \( \mathfrak{g} \) is an ideal of \( \mathfrak{R} \) by the definition of normalizer, and \( \mathcal{Q} \) is ideal of \( \mathfrak{R} \) since \( f(A, [BC]) = f([AB], C) \) for all \( A, B, C \). Thus the lemma is proved.

By a **representation form** on a Lie algebra \( \mathfrak{g} \) we shall mean a bilinear form \( f \) on \( \mathfrak{g} \) for which there is a representation \( S: \mathfrak{g} \to S_\mathfrak{g} = S(\mathfrak{g}) \) of \( \mathfrak{g} \) such that \( f(x, y) = \text{tr}(S_x S_y) \) for all \( x, y \) in \( \mathfrak{g} \).

**Theorem 1.** Let \( \mathfrak{g} \) be a Lie algebra of characteristic \( p \) with a non-degenerate representation form \( f \). Then \( \mathfrak{g} \) is restricted.

**Proof.** Let \( S \) be a representation of \( \mathfrak{g} \) giving rise to \( f \). If \( S_x = 0 \) then \( f(x, \mathfrak{g}) = 0 \), so \( x = 0 \). Hence \( S \) is an isomorphism, so it suffices to prove that the image \( S_\mathfrak{g} \) of \( \mathfrak{g} \) is restricted. For any \( A \) and \( C \) in \( S_\mathfrak{g} \),

\[
[AC^p] = \cdots [AC] \cdots C] \in S_\mathfrak{g}.
\]

Presented to the Society, November 21, 1959 under the title *On Lie algebras of Killing-Cartan-Seligman type*; received by the editors August 17, 1959.

1 This research was supported by the Office of Naval Research.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Hence by the lemma, there is a $B$ in $S_8$ such that $[AC^p] = [AB]$ for all $A$ in $S_8$. Since a Lie algebra is restricted if the $p$th power of every inner derivation is inner, it follows that $S_8$ and $\mathfrak{g}$ are restricted.

We shall write $R$ for the (right) adjoint representation.

**Theorem 2.** Let $\mathfrak{g}$ be a simple Lie algebra over an algebraically closed field of characteristic $p$ with a Cartan subalgebra $\mathfrak{h}$. Suppose that $\mathfrak{g}$ has a nondegenerate representation form $f$. Then for $h$ in $\mathfrak{h}$, $R_h^p = 0$ implies $h = 0$.

**Proof.** Let $S$ be a representation of $\mathfrak{g}$ giving rise to $f$. We assume, without loss of generality, that $S$ is irreducible, of degree $m$. By a result of Zassenhaus [1, p. 7], $\mathfrak{h}$ is abelian, so we can put the matrices $S_8$ in simultaneous triangular form. Now suppose that $h \in \mathfrak{h}$ and $R_h^p = 0$. Then for any $a$ in $\mathfrak{g}$,

$$[S_a S_h^p] = [\cdots [S_a S_h] \cdots S_h] = S_{(\cdots (ah) \cdots )} = S_0 = 0.$$ 

Therefore by Schur’s Lemma, $S_h^p$ is a scalar matrix. Suppose the diagonal of $S_h$ is $(\alpha_1, \cdots, \alpha_m)$. Then the diagonal of $S_h^p$ is $(\alpha_1^p, \cdots, \alpha_m^p)$. Since $S_h^p$ is a scalar, $\alpha_1^p = \cdots = \alpha_m^p$, so $\alpha_1 = \cdots = \alpha_m$. Now if $k$ is any element of $\mathfrak{h}$, and the diagonal of $k$ is $(\beta_1, \cdots, \beta_m)$ then the diagonal of $S_h S_k$ is $(\alpha_1 \beta_1, \cdots, \alpha_m \beta_m)$, so $\text{tr}(S_h S_k) = \alpha_1 \text{tr} S_k$.

But $\text{tr} S_k = 0$ since $S_k \subseteq [S_8 S_8]$. Thus $f(h, k) = 0$ for all $k$ in $\mathfrak{h}$. But any invariant form which is nondegenerate on $\mathfrak{g}$ is nondegenerate on any Cartan subalgebra. Hence $h = 0$ and the theorem is proved.

Now suppose that the hypotheses of Theorem 2 hold. Then $\mathfrak{g}$ must be centerless, and by Theorem 1, $\mathfrak{g}$ is restricted. Hence for any $h$ in $\mathfrak{h}$, there is a unique $h^p$ (necessarily in $\mathfrak{h}$) such that $R_h^p = R(h^p)$. The proofs of the diagonalizability of $\mathfrak{h}$ given in [1, p. 8] and [2, pp. 28, 29] now go through under the present weaker hypotheses. Briefly, the mapping $R_h \rightarrow R_h^p$ is a semilinear mapping of the space of diagonalizable $R_h$ with $h$ in $\mathfrak{h}$. This mapping is one-to-one by Theorem 2, and therefore, by a simple dimension argument, is onto. Hence every diagonalizable $R_h$, $k$ in $\mathfrak{h}$, is the $p$th power of a diagonalizable $R_h$, $h$ in $\mathfrak{h}$. But for any $h$ in $\mathfrak{h}$, some power $R_h^{p*}$ is diagonalizable, and so there is a diagonalizable $R_h$, $k$ in $\mathfrak{h}$, with $R_h^{p*} = R_h^{p*}$. It follows from Theorem 2 that $R_h = R_h$. Hence the following result holds.

**Theorem 3.** Let $\mathfrak{g}$ be as in Theorem 2. Then for every $h$ in $\mathfrak{h}$, $R_h$ acts diagonally on $\mathfrak{g}$, that is, $xh = \alpha(h)x$ for every root $\alpha$ and every $x$ in the root space $\mathfrak{g}_\alpha$.

---

[1] It is actually sufficient to assume that $\mathfrak{g} = 0$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Theorems 2 and 3 generalize results of Jacobson (see [1, pp. 7–8]). It is possible to generalize similarly the results of Jacobson stated as Theorems 4.1 and 4.2 of [1]. Indeed, suppose that the hypotheses of Theorem 2 above are satisfied, and let \( U \) be a representation giving rise to the nondegenerate form \( f \). We suppose without loss of generality that \( U \) is irreducible. Then for any \( x \) in \( \mathfrak{g} \), by Schur’s Lemma, \( U^p_\mathfrak{g} - U(x^p) \) is a scalar matrix, and the trace of this scalar matrix is zero. Thus for any \( e_a \) and \( h \),

\[
\text{tr}(U(e_a)^p U_h) = \text{tr}(U(e_a^p) U_h),
\]

and the proof of Theorem 4.1 given in [1, pp. 11–12] now goes through under the present hypotheses. Now if a nonzero root \( \alpha \) and elements \( e_\alpha \) in \( \mathfrak{g}_\alpha \) and \( e_{-\alpha} \) in \( \mathfrak{g}_{-\alpha} \) are given then \( e_\alpha = e_{-\alpha} = 0 \), so \( U(e_\alpha)^p \) and \( U(e_{-\alpha})^p \) are scalar matrices of trace zero. Thus for a suitable linear functional \( \lambda \) on \( \mathfrak{g} \), by adding the scalar matrix \( \lambda(x) I \) to each \( U \), we obtain a representation \( U' \) for which \( U'(e_\alpha)^p = U'(e_{-\alpha})^p = 0 \), and \( U' \) gives rise to the same trace form as \( U \). With this change, the proof of Theorem 4.2 given in [1] is valid under the present hypotheses. Thus all roots are nonisotropic, that is, if \( \alpha \) is a nonzero root and \( f(h_\alpha, h) = \alpha(h) \) for all \( h \) in \( \mathfrak{g} \), then \( \alpha(h_\alpha) \neq 0 \). For \( p > 3 \), this last result also follows from Theorem 3 by a theorem of Kaplansky [3, p. 165]. We summarize the results of this paragraph in the following theorem.

**Theorem 4.** Let \( \mathfrak{g} \) be as in Theorem 2 and let \( \alpha \) be a nonzero root. Then \( \alpha \) is nonisotropic, and if \( e_\alpha \in \mathfrak{g}_\alpha \), \( e_\alpha^p = 0 \).

As noted in [3], Seligman makes no further use of the fact that his trace form arises from a restricted representation. Thus the result stated in our introduction holds, that is, Seligman’s Theorem 16.2 [1, p. 77] remains valid without the hypothesis of restrictedness of \( \mathfrak{g} \) and its representation.

In particular it follows that the invariant forms of the algebras \( \mathfrak{g}_3 \) and \( \mathfrak{g}(\mathfrak{g}, \delta, f) \) given in [4] do not arise from a representation.

**References**


**Yale University**