CONDITIONS FOR THE POWER ASSOCIATIVITY OF ALGEBRAS

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In this paper we extend the results of Albert [1] and Kokoris [2; 3] to obtain conditions for the power associativity of algebras over arbitrary fields in the noncommutative case. The following results of Albert [1] are used extensively in the development of the theory.

Lemma 1. Let A be an algebra over a field of characteristic p, p ≠ 2, where \( x^ax^b = x^{a+b} \) for \( a + b < n, n \geq 4, x \in A \). Then for \( c = 1, 2, \ldots, n-1 \) it follows that \( [x^{n-c}, x^c] = c[x^{n-1}, x], n[x^{n-1}, x] = 0 \) and for \( (n, p) = 1 \) also \( x^{n-c}x^c = x^cx^{n-c} \).

Lemma 2. Let A be an algebra over a field of characteristic p, p ≠ 2, 3 or 5, where \( x^ax^b = x^{a+b} \) for \( a + b < n, n \geq 5, x \in A \). Then for \( c = 1, 2, \ldots, n-1 \) it follows that \( x^{n-c}x^c = x^n \) for \( (n, p) = 1 \) and \( x^{n-c}x^c = x^{n-1}x + ((c-1)/2)[x^{n-1}, x] \) for all \( n \).

Lemma 3. Let A be an algebra over a field of characteristic p, p ≠ 2, where \( x^ax^b = x^{a+b} \) for \( a + b < n = kp^r, k \neq 1, (k, p) = 1, r \) any positive integer, \( x \in A \). Then \([x^{n-1}, x] = 0\).

Proof. We show that \([x^{n-k}, x^k] = 0\), then by Lemma 1 have \( k[x^{n-1}, x] = 0 \) and \((k, p) = 1\) yields the desired result. \( x^{n-k}x^k = x^{k(p^r-k)x^k} = (x^k)^{p^r-1}x^k = x^k(x^k)^{p^r-1} = x^kx^{n-k} \), using the assumption that \( k \neq 1 \) and that powers with less than \( n \) factors associate.

In the results that follow the reader will easily note that the results of [1; 2; 3] for commutative algebras are consequences of the general theorems.

Case 1, \( p \neq 2, 3 \) or 5.

Theorem 1. Let A be an algebra over a field of characteristic p, \( p \neq 2, 3 \) or 5, and such that for all \( x \in A \) and all positive integers r, (i) \( x^2x = xx^2 \), (ii) \( x^3x = x^2x^2 \), (iii) \( x^{p^r-1}x = xx^{p^r-1} \). Then A is power associative.
PROOF. We observe from Lemma 1 that $x^4x = xx^3$ so that all powers with less than five factors associate. Assuming associativity for all powers with less than $n$ factors, $n \geq 5$, we consider three cases:

(i) For $(n, p) = 1$, Lemma 2 yields $x^{n-5}x = x^n$.

(ii) For $n = p^e$, hypothesis (iii) and Lemma 2 yield the result.

(iii) For $n = kp^e$, $k \neq 1$, $(k, p) = 1$, Lemma 3 yields $[x^{n-1}, x] = 0$ and Lemma 2 then gives the result.

CASE 2, $p = 5$. From Albert [1] the identities $x^2x = xx^2$ and $x^3x = x^2x^2$ imply, for $3 \leq a + b + c < n$,

$$3(x^{n-a}x^a + x^{n-b}x^b + x^{n-c}x^c + x^{n-(a+b+c)}x^{a+b+c})$$

$$= 4(x^{n-a-b}x^a+b + x^{n-a-c}x^a+c + x^{n-b-c}x^b+c) - (a + b + c)[x^{n-1}, x].$$

Assuming associativity for powers with less than $n$ factors and setting $a = b = 1$, $c = 2$ in (2.1) yields

$$x^{n-4}x^4 = x^{n-3}x^3 + 2x^{n-2}x^2 + 3x^{n-1}x + 2[x^{n-1}, x].$$

Under the same associativity conditions, letting the triple $(a, b, c)$ have the values $(1, 1, 3), (1, 1, 4)$ and $(2, 2, 2)$ we obtain the following three identities which combine with (2.2) to yield (2.3).

$$x^{n-5}x^5 = x^{n-4}x^4 + 4x^{n-3}x^3 + 3x^{n-2}x^2 + 3x^{n-1}x,$n=5, it follows that $x^{n-c}x^c = x^n$ for $(n, 5) = 1$ and $x^{n-c}x^c = x^{n-1}x + ((c-1)/2)[x^{n-1}, x]$ for $c < n$.

**Lemma 4.** Let $A$ be an algebra over a field of characteristic 5 and let $x^2x = xx^2$, $x^3x = x^2x^2$, $x^4x = xx^4$, $x^5x = x^4x^2$ for $x \in A$. Then $x^{a+b} = x^a x^b$ for $a + b < 7$, and assuming associativity in powers of less than $n$ factors, $n \geq 7$, it follows that $x^{n-c}x^c = x^n$ for $(n, 5) = 1$ and $x^{n-c}x^c = x^{n-1}x + ((c-1)/2)[x^{n-1}, x]$ for $c < n$.

**Proof.** By Lemma 1, $x^4x = xx^3$ and a substitution of 5 for $n$ in (2.2) with Lemma 1 yields $x^3x = x^4x$ so associativity holds for all powers with three, four or five factors. Lemma 1 gives commutativity of sixth powers, and a substitution of 6 for $n$ in (2.1) with $a = b = c = 1$ yields $2x^4x^2 = 4x^5x + 3x^2x^3$ which, together with the hypothesis $x^4x = x^4x^3$, establishes associativity in sixth powers. The case $c = 1$ of $x^{n-c}x^c = x^{n-1}x + ((c-1)/2)[x^{n-1}, x]$ is trivial and $c = 2$ holds by (2.3). Assuming the validity for $c = 1, 2, \ldots, k-1$ and letting $a = k-2, b = c = 1$ in (2.1) we find $x^{n-k}x^k = x^{n-k+1}x^{k-1} + 4x^{n-k+2}x^{k-2} + 3x^{n-2}x^2 + 3x^{n-1}x + (k/2)[x^{n-1}, x] = x^{n-1}x + ((k-1)/2)[x^{n-1}, x]$. Then for $(n, 5) = 1$, $x^{n-c}x^c = x^n$ follows by Lemma 1.
Theorem 2. Let $A$ be an algebra over a field of characteristic 5 and let
\[ x^2x = xx^2, \quad x^3x = x^2x^2, \quad x^4x = x^3x^2 \quad \text{and} \quad x^{n-1}x = xx^{n-1} \] for all $x \in A$ and all positive integers $r$. Then $A$ is power associative.

Proof. Following the pattern of proof of Theorem 1, an obvious induction in three cases using the hypotheses and Lemmas 3 and 4 yields the proof.

Case 3, $p = 3$. In the case $p = 3$ we make the restriction that the base field is not the prime field and proceed in the same fashion. Letting $a = k - 1$ and $b = c = 1$ in (2.1) we find for $1 < k < n - 1$
\[ x^{n-k}x^k = x^{n-2}x^2 + ((k + 1)/2)[x^{n-1}, x]. \] (3.1)

Thus to get a result corresponding to Lemmas 2 and 5 we relate $x^{n-2}x^2$ and $x^{n-1}x$ in terms of the commutator $[x^{n-1}, x]$. A substitution of $x + \lambda y$ for $x$ in $xx^4 - x^2x^3 = 0$ yields a polynomial equation in $\lambda$, $A\lambda + Bx^2 + Cx^3 + Dx^4 = 0$, and since the base field has at least four non-zero elements $A = 0$. This is just
\[ x[x^3y + (x^2y)x + ((xy + yx)x)x] + yx^4 = x^2[x^2y + (xy + yx)x] + (xy + yx)x^2. \]

Assuming associativity for all powers with less than $n$ factors, $n \geq 6$, and setting $y = x^{n-4}$ we find,
\[ x^{n-4}x^4 = 2x^{n-3}x^3 + 2xx^{n-1}. \]

Now putting $a = b = 2$, $c = 1$, and $a = 3$, $b = c = 1$ in (2.1) and combining yields,
\[ x^{n-4}x^4 = 2x^{n-3}x^3 + 2x^{n-2}x^2. \]

The last two identities and the definition of the commutator yield
\[ x^{n-2}x^2 = xx^{n-1} = x^{n-1}x + 2[x^{n-1}, x]. \] (3.2)

Combining (3.1) and (3.2) we have for $1 < k < n - 1$
\[ x^{n-k}x^k = x^{n-1}x + ((k - 1)/2)[x^{n-1}, x]. \] (3.3)

Lemma 5. Let $A$ be an algebra over a field of characteristic 3, not the prime field, and let $x^2x = xx^2, x^3x = x^2x^2, x^4x = x^3x^2$ for all $x \in A$. Then $x^ax^b = x^{a+b}$ for $a + b < 6$ and assuming that all powers with less than $n$ factors associate, $n \geq 6$, it follows that $x^{n-c}x^c = x^n$ for $(n, 3) = 1$ and $x^{n-c}x^c = x^{n-1}x + ((c-1)/2)[x^{n-1}, x]$ for $c < n$.

Proof. By Lemma 1, with $n = 4$ and 5, and the hypotheses we have associativity for powers with less than 6 factors. Identity (3.3) is
\[ x^{n-c}x^c = x^{n-1}x + ((c-1)/2) [x^{n-1}, x] \] for \( c = 2, 3, \ldots, n-2 \) and the case \( c = 1 \) is trivial. For \( c = n - 1 \) we observe that 
\[ x^{n-(n-1)}x^{n-1} = x^{n-1}x - [x^{n-1}, x] \] and use \( n[x^{n-1}, x] = 0 \) to write 
\[ xx^{n-1}x - [x^{n-1}, x] + (n/2)[x^{n-1}, x] = x^{n-1}x - (((n-1)-1)/2)[x^{n-1}, x] \] as desired. \( x^{n-c}x^c = x^n \) for \( (n, 3) = 1 \) then follows from Lemma 1.

**Theorem 3.** Let \( A \) be an algebra over a field of characteristic 3, not the prime field, and let \( x^3x = x^2x^2, x^4x = x^3x^2 \) and \( x^{3r-1}x = xx^{3r-1} \) for all \( x \in A \) and all positive integers \( r \). Then \( A \) is power associative.

**Proof.** As before, the results follow from the hypotheses by Lemmas 3 and 5, using an induction in three cases.

**Case 4, \( p = 2 \).** In this case we make the restriction that the base field is not the prime field and proceed in a manner somewhat similar to that of Kokoris [3]. We substitute \( x+\lambda y \) for \( x \) in \( x^2x - xx^2 = 0 \) to obtain \( A\gamma + B\gamma^2 = 0 \) where \( A = (xy+yx)x + x^2y - [x(xy+yx)+yx^2] \). Since the base field has at least two nonzero elements, \( A = 0 \), and letting \( y = x^a \) and assuming associativity for powers with less than \( a+2 \) factors we obtain,

\[ x^2x^a = x^ax^2. \]

A substitution of \( x+\lambda y \) for \( x \) in \( x^3x - x^2x^2 = 0 \) gives the polynomial 
\[ A\gamma + B\gamma^2 + C\gamma^3 = 0 \] where \( B = 0 = (x^2y)y + ((xy+yx)x)y + ((xy+yx)y)x + (y^2x)x - [x^2y^2 + (xy+yx)(xy+yx) + y^2x^2] \). Setting \( y = x^a \) in \( B = 0 \), assuming associativity for powers with less than \( 2a+2 \) factors, and using (4.1) yields the first of the following identities. The second follows from \( xx^2 - x^2x^2 = 0 \) by a parallel argument.

\[ x^{a+2}x^a = x^{2a+1}x, \quad x^a x^{a+2} = xx^{2a+1}. \]

Replacing \( y \) by \( x^a + x^{n-(a+2)} \) in \( B = 0 \) and assuming associativity for powers with less than \( n \) factors we obtain the first of the following results. The second is the parallel identity.

\[ x^{n-a}x^a = x^{a+2}x^{n-(a+2)}, \quad x^a x^{n-a} = x^{n-(a+2)}x^{a+2}. \]

From (4.3) we have immediately,

\[ x^{n-a}x^a + x^a x^{n-a} = x^{n-(a+2)}x^{a+2} + x^{a+2}x^{n-(a+2)} \]

and by a simple induction (4.3) also yields for \( 0 \leq 4t \leq n - 2 \),

\[ x^{n-1}x = x^{n-(4t+1)}x^{4t+1}. \]

**Lemma 6.** Let \( A \) be an algebra over a field of characteristic 2, not the prime field, and let \( x^ax^b = x^{a+b} \) for \( a+b < n, n \geq 5 \). Then \( x^{n-2}x^2 = x^{n-b}x^b \) for \( b \) even, \( 0 < b < n \), and \( x^{n-1}x = x^{n-a}x^a \) for \( n \) odd and all \( a, 0 < a < n \).
Proof. From (4.1) and (4.4) we obtain \( x^{n-b}x^b = x^b x^{n-b} \) for all even \( b \). Then (4.3) yields \( x^{n-c}x^c = x^{n-(c+2)}x^{c+2} \) for all even \( c \) and by induction \( x^{n-2}x^2 = x^{n-2b}x^b \) for all even \( b, 0 < b < n \). When \( n \) is odd, \( n-1 \) is even so the above argument yields \( x^{n-1}x = xx^{n-1} \) which extends by (4.4) to \( x^{n-c}x^c = x^n x^{-c} \) for odd \( c, 0 < c < n \). By (4.3) \( x^{n-1}x = x^n x^{-1} \). For even \( c \) we know that \( x^{n-c}x^c = x^{n-2}x^2 = x^2 x^{n-2} \), but \( n \) is odd, \( n-2 \) is odd and \( x^2 x^{n-2} = x^{n-1}x \) by our last result, completing the proof.

Lemma 7. Under the hypotheses of Lemma 6, \( x^{n-1}x = x^{n-a}x^a \) for a odd, \( 0 < a < n, n \) even, and also \( x^{n-1}x = x^{n-a}x^a \) for all \( a, 0 < a < n, n = 2k \) with \( k \) odd and \( k > 1 \).

Proof. We consider two cases, \( n = 2r \) and \( n = 2r k, k \) odd, \( k > 1 \). In the former case the hypothesis \( x^3x = xx^3 \) gives \( x^{n-1}x = x^{n-a}x^a \) for a odd, \( r < 3 \), so we consider only \( r \geq 3 \). Letting \( t = 2^{r-3} \) in (4.5) and \( a = 2^{r-1} - 1 \) in the second relation of (4.2) we obtain \( x^{2r-1}x = xx^{2r-1} \). This extends by (4.3) and (4.4) to \( x^{2r-1}x = x^{2r-a}x^a \) for a odd, \( 0 < a < n = 2r \). In the latter case we note that \( x^{2k-1}x^{2k} = (x^k)^{2r-1}x^k = (x^k)(x^k)^{2r-1} = x^k x^{2k-1} \). Using (4.4) this is extended to \( x^{n-1}x = x^{n-a}x^a \) for all odd \( a \) less than \( n \). To establish the second conclusion of the lemma we exhibit an even \( b, 0 < b < n = 2k \) such that \( x^{n-1}x = x^{n-b}x^b \), then Lemma 6 and the above complete the proof. \( k-1 \) serves as this \( b \) as is seen by substitution of \( k-1 \) for \( a \) in the first of (4.2).

Theorem 4. Let \( A \) be an algebra over a field of characteristic 2, not the prime field, and let \( x^2x = xx^2, x^3x = x^2x^2 = xx^3 \) and \( x^{2r-1}x = x^{2r-1}x^{2r-1} \) for all \( x \in A \) and all integers \( r > 2 \). Then \( A \) is power associative.

Proof. By hypotheses we have associativity for all powers with less than five factors. Assuming associativity for powers with less than \( n \) factors, \( n \geq 5 \), consider the following cases:

(i) \( n \) odd or \( n = 2k \) for \( k \) odd, \( k > 1 \); Lemma 6 or 7 establishes the induction.

(ii) \( n = 2r \); hypothesis \( x^{2r-1}x^{2r-1} = x^{2r-1}x \) yields \( x^{n-1}x = x^{n-b}x^b \) for the even \( b = 2r-1 \) and as in the preceding we have associativity for powers with \( n \) factors.

(iii) \( n = 2rk \) for \( k \) odd, \( k > 1 \); \( x^{2r-1}kx^{2r-1}k = (x^k)^{2r-1}(x^k)^{2r-1} = x^{2r-1}k \) where \( 2r-1 \) is even and \( k \) is odd so that, as above, the induction is completed.

An Example. Examples which show that the conditions of these theorems cannot be weakened appear in Albert [1] and Kokoris [3] with the exception of the hypothesis \( x^2x = xx^2 \) in all cases and the hypothesis \( x^{r-1}x = xx^{r-1} \) in Theorems 1, 2 and 3. The first example is easily constructed after Albert [1] and that for the second set of
conditions is again very similar to one of Albert [1] but we outline the construction since the computation is rather involved.

Let $A$ be the algebra with basis $\{a, a^2, \ldots, a^{p^n-1}, aa^{p^n-1}, a\}$ over a field $F$ of characteristic $p$, $p \neq 0$ or 2 and $p^n > 5$. Define a product by $a^i a^j = a^{s+t}$ for $s+t < p^n$, $a^i a^j = a^{p^n-1}a + 2^{-1}(t-1)a^{p^n-1}, a$ for $s+t = p^n$ and requiring any product with more than $p^n$ factors to be zero. Albert [1] has shown that for $p \neq 3, 5$ this algebra satisfies the hypotheses of Theorem 1 except $x^{p^n-1} = xx^{p^n-1}$ for $i < n$, and the extension to Theorems 2 and 3 is straightforward. For the remaining condition we write the general element of $A$ as

$$x = \sum_{i=1}^{p^n-1} \lambda_i a^i + \lambda_{p^n} a^{p^n-1}a + \lambda_{p^n+1}aa^{p^n-1}$$

Making use of the multinomial expansion

$$x^{p^n-1} = \sum_{k=p^n-1}^{p^n-1} \left[ \sum_{e} \frac{(p^n-1)!}{(e_i)!} \prod \lambda_i^{e_i} \right] a^k + ta^{p^n-1}a + ya^{p^n-1}$$

where $k = \sum ie_i$, $e_i$ are the elements of the partition $e$ of $p^n - 1$, and the values of $t$ and $y$ will not concern us. Since all powers with less than $p^n$ factors associate we may write

$$x^{p^n-1}x = \sum_{m} \left[ \sum_{f} \frac{p^n!}{(f_i)!} \prod \lambda_i^{f_i} \right] a^m + \sum_{k,j} \left[ \sum_{e} \frac{(p^n-1)!}{(e_i)!} \prod \lambda_i^{e_i} \right] a^k a^j$$

where $m = \sum if_i$, $f_i$ are the elements of the partition $f$ of $p^n$, and the $k$ and $e_i$ are as above with $k+j = p^n$. We obtain a similar expression for $xx^{p^n-1}$ and compute the difference

$$x^{p^n-1}x - xx^{p^n-1} = \sum_{e} \sum_{j} \frac{(p^n-1)!}{(e_i)!} \prod \lambda_i^{e_i} \lambda_j[a^{p^n-1}, a].$$

However, $\prod \lambda_i^{e_i} = \prod \lambda_j$ since $f_i = e_i$ for $i \neq j$ and $f_j = e_j + 1$ so that

$$\frac{j}{\prod (e_i!)} = \frac{jj_i}{\prod (f_i!)}$$

and since $\sum jj_i = p^n$ the difference becomes

$$\sum_{e} \frac{p^n!}{(f_i)!} \prod \lambda_i^{f_i} [a^{p^n-1}, a].$$
For any \( r < n \) these coefficients are zero and we have then \( x^{p^r - 1}x = xx^{p^r - 1} \) for all \( r < n \).

**An Application.** A noncommutative Jordan algebra has been defined as an algebra satisfying the identities \( x(yx) = (xy)x \) and \( (x^2y)x = x^2(yx) \) and R. D. Schäfer has proved that any such algebra over a field of characteristic not two is power associative. We extend this result to the characteristic two case when the field is not the prime field. The identities \( x^2x = xx^2 \) and \( x^3x = x^2x^2 = xx^3 \) are trivial and a substitution of \( y = x^{2^n - 2} \) yields \( x^{2^n - 1}x = x^2x^{2^n - 2} \) under the assumption that all powers with less than \( 2^n \) factors associate. Since \( 2^n - 2 \) and \( 2^n - 1 \) are even Lemma 6 yields \( x^{2^n - 1}x = x^{2^n - 1}x^{2^n - 1} \) and the proof of Theorem 4 applies directly.

**Bibliography**


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