CONDITIONS FOR THE POWER ASSOCIATIVITY
OF ALGEBRAS

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In this paper we extend the results of Albert [1] and Kokoris [2; 3] to obtain conditions for the power associativity of algebras over arbitrary fields in the noncommutative case. The following results of Albert [1] are used extensively in the development of the theory.

LEMMA 1. Let $A$ be an algebra over a field of characteristic $p$, $p \neq 2$, where $x^ax^b = x^{a+b}$ for $a+b < n$, $n \geq 4$, $x \in A$. Then for $c = 1, 2, \ldots, n-1$ it follows that $[x^{n-c}, x^c] = c[x^{n-1}, x]$, $n[x^{n-1}, x] = 0$ and for $(n, p) = 1$ also $x^{n-c}x^c = x^cx^{n-c}$.

LEMMA 2. Let $A$ be an algebra over a field of characteristic $p$, $p \neq 2, 3$ or 5, where $x^ax^b = x^{a+b}$ for $a+b < n$, $n \geq 5$, $x \in A$. Then for $c = 1, 2, \ldots, n-1$ it follows that $x^{n-c}x^c = x^n$ for $(n, p) = 1$ and $x^{n-c}x^c = x^{n-1}x + ((c-1)/2)[x^{n-1}, x]$ for all $n$.

LEMMA 3. Let $A$ be an algebra over a field of characteristic $p$, $p \neq 2, 3$ or 5, where $x^ax^b = x^{a+b}$ for $a+b < n = kp$, $k \neq 1$, $(k, p) = 1$, $r$ any positive integer, $x \in A$. Then $[x^{n-1}, x] = 0$.

Proof. We show that $[x^{n-k}, x^k] = 0$, then by Lemma 1 have $k[x^{n-1}, x] = 0$ and $(k, p) = 1$ yields the desired result. $x^{n-k}x^k = x^{k+1}x^{-1}/x^k = (x^k)^{p-1}x^{k} = x^k(x^k)^{p-1} = x^kx^{n-k}$, using the assumption that $k \neq 1$ and that powers with less than $n$ factors associate.

In the results that follow the reader will easily note that the results of [1; 2; 3] for commutative algebras are consequences of the general theorems.

Case 1, $p \neq 2, 3$ or 5.

THEOREM 1. Let $A$ be an algebra over a field of characteristic $p$, $p \neq 2, 3$ or 5, and such that for all $x \in A$ and all positive integers $r$, (i) $x^2x = xx^2$, (ii) $x^3x = x^2x^2$, (iii) $x^{p^r-1}x = xx^{p^r-1}$. Then $A$ is power associative.
Proof. We observe from Lemma 1 that $x^4x = xx^3$ so that all powers with less than five factors associate. Assuming associativity for all powers with less than $n$ factors, $n \geq 5$, we consider three cases:

(i) For $(n, \rho) = 1$, Lemma 2 yields $x^{n-c}x^c = x^n$.

(ii) For $n = \rho^r$, hypothesis (iii) and Lemma 2 yield the result.

(iii) For $n = k\rho^r$, $k \neq 1$, $(k, \rho) = 1$, Lemma 3 yields $[x^{n-1}, x] = 0$ and Lemma 2 then gives the result.

Case 2, $\rho = 5$. From Albert [1] the identities $x^2x = xx^2$ and $x^3x = x^2x^2$ imply, for $3 \leq a + b + c < n$,

\[3(x^{n-a}x^a + x^{n-b}x^b + x^{n-c}x^c + x^{n-(a+b+c)}x^{a+b+c}) = 4(x^{n-a}x^a + x^{n-b}x^b + x^{n-c}x^c + x^{n-(a+b+c)}x^{a+b+c}) - (a + b + c)[x^{n-1}, x].\]

Assuming associativity for powers with less than $n$ factors and setting $a = b = 1$, $c = 2$ in (2.1) yields

\[(2.2) \quad x^{n-2}x^2 = x^{n-1}x + 3[x^{n-1}, x]. \quad \text{for } a = b = 1, \ c = 2 \text{ in } (2.1) \text{ yields} \]

Under the same associativity conditions, letting the triple $(a, b, c)$ have the values $(1, 1, 3)$, $(1, 1, 4)$ and $(2, 2, 2)$ we obtain the following three identities which combine with (2.2) to yield (2.3).

\[
\begin{align*}
x^{n-5}x^6 &= x^{n-4}x^4 + 4x^{n-3}x^3 + 3x^{n-2}x^2 + 3x^{n-1}x, \\
x^{n-6}x^8 &= x^{n-5}x^5 + 4x^{n-4}x^4 + 3x^{n-2}x^2 + 3x^{n-1}x + 3[x^{n-1}, x], \\
x^{n-6}x^6 &= 4x^{n-4}x^4 + 2x^{n-2}x^2 + 3[x^{n-1}, x].
\end{align*}
\]

\[(2.3) \quad x^{n-2}x^2 = x^{n-1}x + 3[x^{n-1}, x] \quad \text{for } n \geq 7.\]

Lemma 4. Let $A$ be an algebra over a field of characteristic 5 and let $x^2x = xx^2$, $x^3x = x^2x^2$, $x^4x = xx^4$, $x^5x = x^4x^2$ for $x \in A$. Then $x^a x^b = x^{a+b}$ for $a + b < 1$, and assuming associativity in powers of less than $n$ factors, $n \geq 7$, it follows that $x^{n-c}x^c = x^n$ for $(n, 5) = 1$ and $x^{n-c}x^c = x^{n-1}x + ((c-1)/2)[x^{n-1}, x]$ for $c < n$.

Proof. By Lemma 1, $x^4x = xx^3$ and a substitution of 5 for $n$ in (2.2) with Lemma 1 yields $x^3x^2 = xx^2$ so associativity holds for all powers with three, four or five factors. Lemma 1 gives commutativity of sixth powers, and a substitution of 6 for $n$ in (2.1) with $a = b = c = 1$ yields $2x^4x^2 = 4x^6x + 3x^3x^3$ which, together with the hypothesis $x^6x = x^4x^2$, establishes associativity in sixth powers. The case $c = 1$ of $x^{n-c}x^c = x^{n-1}x + ((c-1)/2)[x^{n-1}, x]$ is trivial and $c = 2$ holds by (2.3).

Assuming the validity for $c = 1, 2, \cdots, k-1$ and letting $a = k-2$, $b = c = 1$ in (2.1) we find $x^{n-2}x^k = x^{n-k+1}x^{k-1} + 4x^{n-k+2}x^{k-2} + 3x^{n-2}x^2 + 3x^{n-1}x + (k/2)[x^{n-1}, x] = x^{n-1}x + ((k-1)/2)[x^{n-1}, x]$. Then for $(n, 5) = 1$, $x^{n-c}x^c = x^n$ follows by Lemma 1.
Theorem 2. Let $A$ be an algebra over a field of characteristic 5 and let $x^2x = xx^2$, $x^3x = x^2x^2$, $x^5x = x^4x^2$ and $x^{5-r}x = xx^{5-r-1}$ for all $x \in A$ and all positive integers $r$. Then $A$ is power associative.

Proof. Following the pattern of proof of Theorem 1, an obvious induction in three cases using the hypotheses and Lemmas 3 and 4 yields the proof.

Case 3, $p = 3$. In the case $p = 3$ we make the restriction that the base field is not the prime field and proceed in the same fashion. Letting $a = k - 1$ and $b = c = 1$ in (2.1) we find for $1 < k < n - 1$

$$x^{n-k}x^k = x^{n-2}x^2 + ((k + 1)/2)[x^{n-1}, x].$$

Thus to get a result corresponding to Lemmas 2 and 5 we relate $x^{n-2}x^2$ and $x^{n-1}x$ in terms of the commutator $[x^{n-1}, x]$. A substitution of $x + \lambda y$ for $x$ in $xx^4 - x^2x^3 = 0$ yields a polynomial equation in $\lambda$, $A\lambda + B\lambda^2 + C\lambda^3 + D\lambda^4 = 0$, and since the base field has at least four non-zero elements $A = 0$. This is just

$$x[x^3y + (x^2y)x + ((xy + yx)x)x] + yx^4 = x^2[x^2y + (xy + yx)x] + (xy + yx)x^3.$$  

Assuming associativity for all powers with less than $n$ factors, $n \geq 6$, and setting $y = x^{n-4}$ we find,

$$x^{n-4}x^4 = 2x^{n-3}x^3 + 2x^n - 1.$$  

Now putting $a = b = 2, c = 1$, and $a = 3, b = c = 1$ in (2.1) and combining yields,

$$x^{n-4}x^4 = 2x^{n-3}x^3 + 2x^{n-2}x^2.$$  

The last two identities and the definition of the commutator yield

$$x^{n-2}x^2 = xx^{n-1} = x^{n-1}x + 2[x^{n-1}, x].$$

Combining (3.1) and (3.2) we have for $1 < k < n - 1$

$$x^{n-k}x^k = x^{n-1}x + ((k - 1)/2)[x^{n-1}, x].$$

Lemma 5. Let $A$ be an algebra over a field of characteristic 3, not the prime field, and let $x^2x = xx^2$, $x^3x = x^2x^2$, $x^4x = x^3x^2$ for all $x \in A$. Then $x^a x^b = x^{a+b}$ for $a + b < 6$ and assuming that all powers with less than $n$ factors associate, $n \geq 6$, it follows that $x^{n-c} x^c = x^n$ for $(n, 3) = 1$ and $x^{n-c} x^c = x^{n-1}x + ((c - 1)/2)[x^{n-1}, x]$ for $c < n$.

Proof. By Lemma 1, with $n = 4$ and 5, and the hypotheses we have associativity for powers with less than 6 factors. Identity (3.3) is
\[ x^{n-c}x^c = x^{n-1}x + ((c-1)/2)[x^{n-1}, x] \] for \( c = 2, 3, \ldots, n-2 \) and the case \( c = 1 \) is trivial. For \( c = n-1 \) we observe that \( x^{n-(n-1)}x^{n-1} = x^{n-1}x - [x^{n-1}, x] \) and use \( n[x^{n-1}, x] = 0 \) to write \( xx^{n-1} = x^{n-1}x - (n/2)[x^{n-1}, x] = x^{n-1}x - (((n-1)-1)/2)[x^{n-1}, x] \) as desired. \( x^{n-c}x^c = x^n \) for \( (n, 3) = 1 \) then follows from Lemma 1.

**Theorem 3.** Let \( A \) be an algebra over a field of characteristic 3, not the prime field, and let \( x^3x = x^2x^2 \), \( x^4x = x^3x^2 \) and \( x^{3r-1}x = xx^{3r-1} \) for all \( x \in A \) and all positive integers \( r \). Then \( A \) is power associative.

**Proof.** As before, the results follow from the hypotheses by Lemmas 3 and 5, using an induction in three cases.

**Case 4,** \( p = 2 \). In this case we make the restriction that the base field is not the prime field and proceed in a manner somewhat similar to that of Kokoris [3]. We substitute \( x \pm y \) for \( x \) in \( x^2x - xx^2 = 0 \) to obtain \( Ay + By^2 = 0 \) where \( A = (xy + yx)x + x^2y - [x(xy+yx) + yx^2] \). Since the base field has at least two nonzero elements, \( A = 0 \), and letting \( y = x^a \) and assuming associativity for powers with less than \( a+2 \) factors we obtain,

\[ x^2x^a = x^a x^2. \]

A substitution of \( x + \lambda y \) for \( x \) in \( x^2x - xx^2 = 0 \) gives the polynomial \( A\lambda + 2x^3 + C \lambda^3 = 0 \) where \( B = 0 = (x^2y) + ((xy+yx)x)y + ((xy+yx)y)x + (y^2x)x - [x^2y^2 + (xy+yx)(xy+yx) + y^2x^2]. \) Setting \( y = x^a \) in \( B = 0 \), assuming associativity for powers with less than 2\( a+2 \) factors, and using (4.1) yields the first of the following identities. The second follows from \( xx^2 - x^2x^2 = 0 \) by a parallel argument.

\[ x^{a+2}x^a = x^{2a+1}x, \quad x^a x^{a+2} = xx^{2a+1} \]

Replacing \( y \) by \( x^a + x^{a-(a+2)} \) in \( B = 0 \) and assuming associativity for powers with less than \( n \) factors we obtain the first of the following results. The second is the parallel identity.

\[ x^{n-a}x^a = x^{a+2}x^{n-(a+2)}, \quad x^a x^{n-a} = x^{n-(a+2)}x^{a+2}. \]

From (4.3) we have immediately,

\[ x^{n-a}x^a + x^a x^{n-a} = x^{n-(a+2)}x^{a+2} \]

and by a simple induction (4.3) also yields for \( 0 \leq 4t \leq n-2 \),

\[ x^{n-1}x = x^{n-(4t+1)}x^{4t+1}. \]

**Lemma 6.** Let \( A \) be an algebra over a field of characteristic 2, not the prime field, and let \( x^a x^b = x^{a+b} \) for \( a + b < n \), \( n \geq 5 \). Then \( x^{n-2}x^2 = x^{n-2}x^b \) for \( b \) even, \( 0 < b < n \), and \( x^{n-1}x = x^{n-1}x^a \) for \( n \) odd and all \( a, 0 < a < n \).
Proof. From (4.1) and (4.4) we obtain \( x^{n-b}x^b = x^b x^{n-b} \) for all even \( b \). Then (4.3) yields \( x^{n-c}x^c = x^{n-(c+2)}x^{c+2} \) for all even \( c \) and by induction \( x^{n-2}x^2 = x^{n-2}x^b \) for all even \( b \), \( 0 < b < n \). When \( n \) is odd, \( n - 1 \) is even so the above argument yields \( x^{n-1}x = xx^{n-1} \) which extends by (4.4) to \( x^{n-c}x^c = x^c x^{n-c} \) for odd \( c \), \( 0 < c < n \). By (4.3) \( x^{n-1}x = x^n x^{n-1} \). For even \( c \) we know that \( x^{n-c}x^c = x^{n-2}x^2 = x^2 x^{n-2} \), but \( n \) is odd, \( n - 2 \) is odd and \( x^2 x^{n-2} = x^{n-1}x \) by our last result, completing the proof.

Lemma 7. Under the hypotheses of Lemma 6, \( x^{n-1}x = x^n - a x^a \) for an odd, \( 0 < a < n \), \( n \) even, and also \( x^{n-1}x = x^n - a x^a \) for all \( a \), \( 0 < a < n \), \( n = 2k \) with \( k \) odd and \( k > 1 \).

Proof. We consider two cases, \( n = 2r \) and \( n = 2rk \), \( k \) odd, \( k > 1 \). In the former case the hypothesis \( x^2x = xx^2 \) gives \( x^{n-1}x = x^n - a x^a \) for an odd, \( r < 3 \), so we consider only \( r \geq 3 \). Letting \( t = 2r-3 \) in (4.5) and \( a = 2r-1 \) in the second relation of (4.2) we obtain \( x^{2r-1}x = xx^{2r-1} \). This extends by (4.3) and (4.4) to \( x^{2r-1}x = x^{2r-2} x^a \) for an odd, \( 0 < a < n = 2r \). In the latter case we note that \( x^{2k-1}x = (x^k)^{2r-1}x = (x^k)(x^k)^{2r-1} = x^k x^{2r-2} \). Using (4.4) this is extended to \( x^{n-1}x = x^n - a x^a \) for all odd \( a \) less than \( n \). To establish the second conclusion of the lemma we exhibit an even \( b \), \( 0 < b < n = 2k \) such that \( x^{n-1}x = x^n - bx^b \), then Lemma 6 and the above complete the proof.

Theorem 4. Let \( A \) be an algebra over a field of characteristic 2, not the prime field, and let \( x^2x = xx^2 \), \( x^3x = x^2x^2 = xx^3 \) and \( x^{2r-1}x = x^{2r-1} x^{2r-1} \) for all \( x \in A \) and all integers \( r > 2 \). Then \( A \) is power associative.

Proof. By hypotheses we have associativity for all powers with less than five factors. Assuming associativity for powers with less than \( n \) factors, \( n \geq 5 \), consider the following cases:

(i) \( n \) odd or \( n = 2k \) for \( k \) odd, \( k > 1 \); Lemma 6 or 7 establishes the induction.

(ii) \( n = 2r \); hypothesis \( x^{2r-1}x^{2r-1} = x^{2r-1}x \) yields \( x^{n-1}x = x^n - bx^b \) for the even \( b = 2r-1 \) and as in the preceding we have associativity for powers with \( n \) factors.

(iii) \( n = 2rk \) for \( k \) odd, \( k > 1 \); \( x^{2r-1}k x^{2r-1}k = (x^k)^{2r-1} = x^{2r-1} x^{2r-1} \) where \( 2r-1k \) is even and \( k \) is odd so that, as above, the induction is completed.

An Example. Examples which show that the conditions of these theorems cannot be weakened appear in Albert [1] and Kokoris [3] with the exception of the hypothesis \( x^2x = xx^2 \) in all cases and the hypothesis \( x^{2r-1}x = xx^{2r-1} \) in Theorems 1, 2 and 3. The first example is easily constructed after Albert [1] and that for the second set of
conditions is again very similar to one of Albert [1] but we outline
the construction since the computation is rather involved.

Let $A$ be the algebra with basis $\{a, a^2, \ldots, a^{p^n-1}, a^{p^n-1}a, a^{p^n-1}a^2, \ldots\}$
over a field $F$ of characteristic $p$, $p \neq 0$ or 2 and $p^5 > 5$. Define a
product by $a^s a^t = a^{s+t}$ for $s + t < p^n$, $a^s a^t = a^{p^n-1}a + 2^{-1}(t-1) [a^{p^n-1}, a]$ for $s + t = p^n$ and requiring any product with more than $p^n$ factors
to be zero. Albert [1] has shown that for $p \neq 3, 5$ this algebra satisfies
the hypotheses of Theorem 1 except $x^{p^n-1}x = xx^{p^n-1}$ for $i < n$, and the
extension to Theorems 2 and 3 is straightforward. For the remaining
condition we write the general element of $A$ as

$$x = \sum_{i=1}^{p^n-1} \lambda_i a^i + \lambda_{p^n a^{p^n-1}} a + \lambda_{p^n+1 a^{p^n-1}}$$

Making use of the multinomial expansion

$$x^{p^n-1} = \sum_{k=p^n-1}^{p^n-1} \left[ \sum_e \frac{(p^n - 1)!}{\prod (e_i)!} \prod \lambda_i^{e_i} \right] a^k + t a^{p^n-1} a + ya a^{p^n-1}$$

where $k = \sum i e_i$, $e_i$ are the elements of the partition $e$ of $p^n - 1$, and
the values of $t$ and $y$ will not concern us. Since all powers with less
than $p^n$ factors associate we may write

$$x^{p^n-1} x = \sum_{m} \left[ \sum_f \frac{p^n!}{\prod (f_i)!} \prod \lambda_i^{f_i} \right] a^m$$

where $m = \sum i f_i$, $f_i$ are the elements of the partition $f$ of $p^n$, and the
$k$ and $e_i$ are as above with $k + j = p^n$. We obtain a similar expression
for $xx^{p^n-1}$ and compute the difference

$$x^{p^n-1} x - xx^{p^n-1} = \sum_{o} \sum_j \frac{(p^n - 1)!}{\prod (e_i)!} \prod \lambda_i^{e_i} \lambda_j [a^{p^n-1}, a].$$

However, $\prod \lambda_i^{f_i} = \prod \lambda_i^{f_j}$ since $f_i = e_i$ for $i \neq j$ and $f_j = e_j + 1$ so that

$$\frac{j}{\prod (e_i)!} = \frac{j f_i}{\prod (f_i)!}$$

and since $\sum j f = p^n$ the difference becomes

$$\sum_e \frac{p^n!}{\prod (f_i)!} \prod \lambda_i^{f_i} [a^{p^n-1}, a].$$
For any $r < n$ these coefficients are zero and we have then $x^{p^r-1} = xx^{p^r-1}$ for all $r < n$.

An Application. A noncommutative Jordan algebra has been defined as an algebra satisfying the identities $x(yx) = (xy)x$ and $(x^2y)x = x^2(yx)$ and R. D. Schafer has proved that any such algebra over a field of characteristic not two is power associative. We extend this result to the characteristic two case when the field is not the prime field. The identities $x^2x = xx^2$ and $x^3x = x^2x^2 = xx^3$ are trivial and a substitution of $y = x^{2^n-3}$ yields $x^{2^n-1}x = x^2x^{2^n-2}$ under the assumption that all powers with less than $2^n$ factors associate. Since $2^n - 2$ and $2^n - 1$ are even Lemma 6 yields $x^{2^n-1}x = x^{2^n-1}x^{2^n-1}$ and the proof of Theorem 4 applies directly.

Bibliography


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