DIVISIBLE MODULES

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Introduction. Let R be an integral domain with quotient field Q, and let A be a module over R. A is said to be a *divisible* R-module, if $rA=A$ for every $r\neq 0\in R$. An element $x\in A$ is said to be a *torsion element* of A, if there exists $r\neq 0\in R$ such that $rx=0$. The set of torsion elements of A is a submodule of A called the *torsion submodule* of A, and we will consistently denote it by A_T. We will let $E(A)$ denote the *injective envelope* of A (see [3]); and $\mathrm{hd}_R\, A$ will denote the *homological dimension* of A as an R-module.

We will study conditions, some necessary, some sufficient, for the torsion submodule of a divisible module to be a direct summand. These will be related to the condition that $\mathrm{hd}_R\, Q=1$, where Q is the quotient field of R. We will apply these conditions to show that, if R is a Noetherian integral domain in which prime ideals different from zero are maximal, and if D is a divisible module over R, then D is a homomorphic image of an injective R-module and D_T is a direct summand of D. The same conclusions hold, if we merely assume for an arbitrary integral domain that its quotient field is countably generated as a module over the ring.

1. The torsion submodule. It is easy to see that, if C is an injective module over an integral domain R, then its torsion submodule C_T is also an injective R-module, and therefore a direct summand. Namely, any homomorphism of an ideal of R into C_T can be extended to a homomorphism of R into C; but this extension must in fact map R into C_T. The following theorem is a generalization of this fact.

THEOREM 1.1. *Let R be an integral domain and H a homomorphic image of an injective R-module. Then H_T is a direct summand of H.*

PROOF. The mapping of a free R-module F onto an injective R-module C can be extended to a mapping of $E(F)$ onto C. We can thus assume that there exists a torsion-free, divisible R-module U and an epimorphism $f\colon U\to H$. Now H/H_T, being torsion-free and divisible, is a direct sum of R-modules Q_i, where Q_i is isomorphic to Q the quotient field of R. Let S_i be the inverse image of Q_i under the canonical map $H\to H/H_T$. We will prove that H_T is a direct summand of each S_i, and by [2, Lemma 2] this will complete the proof of the theorem.

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Let $y \in S_j - H_T$; then Ry is a torsion-free submodule of S_j. Choose $x \in U$ such that $f(x) = y$. Since U is a vector space over Q, there exists an R-submodule T_j of U such that $T_j \cong Q$ and $x \in T_j$. If $u \neq 0 \in T_j$, there exist r, $s \in R$ such that $ru = sx \neq 0$. Since $rf(u) = sf(x) = sy \neq 0$, we have $f(u) \neq 0$, and thus $f(T_j) \cong T_j \cong Q$.

Since $H = \sum_i S_i$, $f(u) = w + z$, where $w \in \sum_{i \neq j} S_i$ and $z \in S_j$. Hence $rw + rz = rf(u) = sy \in S_j$. Thus $rw \in \sum_{i \neq j} S_i \cap S_j = H_T$. Therefore, $w \in H_T \subset S_j$, and so $f(u) \in S_j$. This shows that $f(T_j) \subset S_j$. Since $f(T_j) \cong Q$, we have $f(T_j) \cap H_T = 0$ and $f(T_j)$ maps onto Q_j under the canonical map $H \to H/H_T$. Thus $S_j = H_T \oplus f(T_j)$; and so H_T is a direct summand of H.

THEOREM 1.2. *Let R be an integral domain with quotient field $Q \neq R$. Suppose that D_T is a direct summand of D for every divisible R-module D. Then* $\mathrm{hd}_R\, Q = 1$.

PROOF. Let A be any R-module and let $E = E(A)$. Since E is an essential extension of A, E/A is a torsion R-module. Let G be any R-module extension of E/A by Q. Since both Q and E/A are divisible, G is also divisible. Clearly E/A is the torsion submodule of G; and thus by assumption E/A is a direct summand of G. Thus $\mathrm{Ext}^1_R(Q, E/A) = 0$ [1, Theorem 14.1.1]. We also have $\mathrm{Ext}^n_R(Q, E) = 0$ for $n > 0$, since E is injective. Therefore, from the exact sequence:

$$\mathrm{Ext}^1_R(Q, E/A) \to \mathrm{Ext}^2_R(Q, A) \to \mathrm{Ext}^2_R(Q, E)$$

we deduce that $\mathrm{Ext}^2_R(Q, A) = 0$. Hence $\mathrm{hd}_R\, Q \leqq 1$. Since Q is not R-projective, $\mathrm{hd}_R\, Q = 1$.

Over an arbitrary integral domain it is not true that D_T is a direct summand for every divisible module D. For by the above theorem this would imply that $\mathrm{hd}_R\, Q = 1$. However, I. Kaplansky has shown (unpublished) that $\mathrm{hd}_R\, Q = 1$ for a valuation ring R if and only if Q is a countably generated R-module.

THEOREM 1.3. *Let R be an integral domain with quotient field $Q \neq R$, and suppose that Q is countably generated as a R-module. Then every divisible R-module D is a homomorphic image of an injective R-module. Thus D_T is a direct summand of D, and* $\mathrm{hd}_R\, Q = 1$.

PROOF. There exists a countable set of generators $\{q_n\}$ for Q over R, and elements $\{a_{n+1}\}$ of R such that $q_1 = 1$ and $a_{n+1} q_{n+1} = q_n$. Let D be a divisible R-module, and let $x \neq 0 \in D$. We define a mapping f from the generators $\{q_n\}$ to D. Let $f(1) = x$; then there exists $x_2 \in D$ such that $a_2 x_2 = x$, and we define $f(q_2) = x_2$. There exists $x_3 \in D$ such that $a_3 x_3 = x_2$, and we define $f(q_3) = x_3$. We continue in this way and

define f on all the generators $\{q_n\}$. It is easily verified that f induces an R-homomorphism from Q into D such that the image contains x. It is now clear that by taking a big enough direct sum G of copies of Q we can define an R-homomorphism of G onto D.

It should be remarked that if R is any integral domain and S a countable, multiplicatively closed subset of R, then it can be easily shown that if F is a countably generated free R-module and f a suitably chosen mapping of F onto R_S, then the kernel of f is free; and thus $\mathrm{hd}_R\, R_S \leqq 1$.

2. $\mathrm{hd}_R\, Q = 1$.

PROPOSITION 2.1. *Let R be an integral domain with quotient field Q such that* $\mathrm{hd}_R\, Q = 1$. *Let H be an R-module. Then the following statements are equivalent*:

(1) $\mathrm{Ext}^1_R(Q/R, H) = 0$.

(2) *Every R-homomorphism from R into H can be extended to an R-homomorphism from Q into H.*

(3) *H is a homomorphic image of an injective R-module.*

PROOF. That (1) implies (2) follows immediately from the exact sequence:

$$\mathrm{Hom}_R(Q, H) \to \mathrm{Hom}_R(R, H) \to \mathrm{Ext}^1_R(Q/R, H).$$

That (2) implies (3) is trivial. That (3) implies (1) follows from the fact that $\mathrm{hd}_R\, Q/R = 1$.

PROPOSITION 2.2. *Let R be an integral domain with quotient field Q such that* $\mathrm{hd}_R\, Q = 1$. *Let H be an R-module. Then*:

(1) *If H is a homomorphic image of an injective R-module, so is H_T.*

(2) *If B is a submodule of H and if B and H/B are homomorphic images of injective modules, then so is H.*

PROOF.

(1) If H is a homomorphic image of an injective R-module, then H_T is a direct summand of H by Theorem 1.1. Hence H_T is also a homomorphic image of an injective R-module.

(2) Suppose that B and H/B are homomorphic images of injective R-modules. We have an exact sequence:

$$\mathrm{Ext}^1_R(Q/R, B) \to \mathrm{Ext}^1_R(Q/R, H) \to \mathrm{Ext}^1_R(Q/R, H/B).$$

By Proposition 2.1 the two end modules are zero, and thus $\mathrm{Ext}^1_R(Q/R, H) = 0$. Hence by Proposition 2.1 again, H is a homomorphic image of an injective R-module.

DEFINITION. Let B be a module over an integral domain. Then we will say that B is *h-reduced*, if B has no nonzero submodules which are homomorphic images of injective modules.

COROLLARY 2.3. *Let A be a module over an integral domain R with quotient field Q such that* $\mathrm{hd}_R\, Q=1$. *Then A has a unique largest submodule H which is a homomorphic image of an injective R-module, and A/H is h-reduced.*

PROOF. Let H be the sum of all submodules of A which are homomorphic images of injective R-modules. It is clear that H is the unique largest submodule of A which is a homomorphic image of an injective R-module. Suppose that B/H is a homomorphic image of an injective R-module, where B is a submodule of A containing H. Then by Proposition 2.2 B is a homomorphic image of an injective R-module. Therefore, $B=H$ and $B/H=0$.

PROPOSITION 2.4. *Let R be an integral domain with quotient field Q such that* $\mathrm{hd}_R\, Q=1$. *Let D be a divisible module over R, and let H be a submodule of D_T such that H is a homomorphic image of an injective R-module. Then D_T is a direct summand of D if and only if D_T/H is a direct summand of D/H.*

PROOF. Suppose that D_T is a direct summand of D, and let S be a complementary summand of D_T in D. Then $D/H\cong D_T/H\oplus S$, and since D_T/H is the torsion submodule of D/H, D_T/H is a direct summand of D/H. Conversely, suppose that $D/H=D_T/H\oplus G/H$, where G is a submodule of D containing H. Now G/H is torsion-free and divisible, hence injective. Thus by Proposition 2.2 and Theorem 1.1 H is a direct summand of G. Let L be a complementary summand of H in G. Then it is clear that $D=D_T\oplus L$.

PROPOSITION 2.5. *Let R be an integral domain with quotient field Q, and let T be an h-reduced torsion R-module. Then* $\mathrm{Ext}^1_R(Q, T)=0$ *if and only if* $T\cong\mathrm{Ext}^1_R(Q/R, T)$.

PROOF. Since $\mathrm{Hom}_R(Q, T)=0$, we have an exact sequence:

$$0\to \mathrm{Hom}_R(R, T)\to \mathrm{Ext}^1_R(Q/R, T)\to \mathrm{Ext}^1_R(Q, T)\to 0.$$

It follows that if $\mathrm{Ext}^1_R(Q, T)=0$, then $T\cong\mathrm{Ext}^1_R(Q/R, T)$. Conversely, if $T\cong\mathrm{Ext}^1_R(Q/R, T)$, then the above exact sequence shows that $\mathrm{Ext}^1_R(Q, T)$ is a torsion module. However, $\mathrm{Ext}^1_R(Q, T)$ is torsion-free, and thus $\mathrm{Ext}^1_R(Q, T)=0$.

COROLLARY 2.6. *Let R be an integral domain with quotient field Q.*

Then the torsion submodule of a divisible R-module is always a direct summand if and only if the following two conditions hold:

(1) $\mathrm{hd}_R\, Q=1$.

(2) $T\cong \mathrm{Ext}_R^1(Q/R, T)$, *whenever T is an h-reduced, torsion, divisible R-module.*

PROOF. The necessity follows from Theorem 1.2 and Proposition 2.5; the sufficiency follows from Corollary 2.3 and Propositions 2.4 and 2.5.

PROPOSITION 2.7. *Let R be an integral domain with quotient field Q such that* $\mathrm{hd}_R\, Q=1$ *and* gl. dim. $R\leqq 2$. *Let S be any torsion-free R-module. Then* $\mathrm{hd}_R\, S\leqq 1$. *Thus if A is an R-module such that* A_T *is a homomorphic image of an injective R-module, then* A_T *is a direct summand of A.*

PROOF. Let B be any R-module. Then from the exact sequence:

$$0\to S\to Q\otimes_R S\to Q/R\otimes_R S\to 0$$

we derive the exact sequence:

$$\mathrm{Ext}_R^2(Q\otimes_R S, B)\to \mathrm{Ext}_R^2(S, B)\to \mathrm{Ext}_R^3(Q/R\otimes_R S, B).$$

Since $\mathrm{hd}_R\, Q=1$ and gl. dim. $R\leqq 2$, the two end modules are zero Thus $\mathrm{hd}_R\, S\leqq 1$, and the rest of the theorem follows immediately.

2. Krull dimension = 1.

Throughout this section R will be a Noetherian integral domain with the property that nonzero prime ideals are maximal. We will let Q be the quotient field of R and $K=Q/R$.

DEFINITION. Let A be an R-module and M a prime ideal of R. We will say that A is *M-primary*, if for any $x\neq 0\in A$, the order ideal of x is an M-primary ideal. If B is any R-module, and A is the set of all elements of B whose order ideal is M-primary (together with the element 0), then A is an M-primary R-module which we will call the *M-primary component* of B.

LEMMA 3.1. *Let B be any torsion R-module. Then B is the direct sum of its M-primary components, M ranging over the prime ideals of R. Furthermore,* $B\otimes_R R_M$ *is the M-primary component of B.*

PROOF. Let $\{M_a\}$ be the collection of nonzero prime ideals of R. By [3, Theorem 3.3] $E(B)=\sum_a\oplus E_a$, where E_a is the M_a-component of $E(B)$. Let $B_a=B\cap E_a$; then B_a is the M_a-component of B. Let $x\neq 0\in B$; then $x=x_1+\cdots+x_n$, where $x_i\in E_i$. Now $\bigcap_{i=2}^n M_i\not\subset M_1$;

hence there exists $s \in \bigcap_{2}^{n} M_i$ such that $s \notin M_1$. Then there exists an integer $k>0$ such that $s^k x_i = 0$ for $i=2, \cdots, n$. Hence $s^k x = s^k x_1$. There are elements $m \in M$ and $t \in R$ such that $1 = m + ts^k$. There is an integer $q>0$ such that $m^q x_1 = 0$. Since $1 = m^q + rs^k$, $r \in R$, we have $x_1 = rs^k x_1 = rs^k x \in B_1$. Similarly $x_i \in B_i$ for $i=2, \cdots, n$. Thus $B = \sum_a \oplus B_a$.

Let M_v be a prime ideal of R. Clearly $B_a \otimes_R R_{M_v} = 0$, if $M_a \neq M_v$. Thus $B \otimes_R R_{M_v} = B_v \otimes_R R_{M_v}$. It is easily seen that the canonical map $B_v \to B_v \otimes_R R_{M_v}$ is an epimorphism. However, since B_v is M_v-primary, the kernel of this map is zero. Thus $B_v = B_v \otimes_R R_{M_v}$, and so $B \otimes_R R_{M_v} = B_v$.

LEMMA 3.2. $\mathrm{hd}_R\, Q = 1$.

PROOF. It is sufficient to prove that $\mathrm{hd}_R\, K = 1$. By Lemma 3.1 $K = \sum_a \oplus K_{M_a}$; thus it is sufficient to prove that $\mathrm{hd}_R\, K_{M_v} = 1$. For this it is sufficient to prove that if D is any divisible R-module, then $\mathrm{Ext}_R^1(K_{M_v}, D) = 0$. Let A be any extension of D by K_{M_v}; then A is a divisible R-module. We have $D = \sum_a \oplus D_{M_a}$ and $A = \sum_a \oplus A_{M_a}$. Clearly $D_{M_a} = D \cap A_{M_a}$. Hence for $a \neq v$, we have $D_{M_a} = A_{M_a}$. Thus we have an exact sequence:

$$0 \to D_{M_v} \to A_{M_v} \to K_{M_v} \to 0,$$

and all of the modules and mappings of this sequence are R_{M_v}-modules and mappings. Thus we can assume that R is a local ring with a single nonzero prime ideal M.

Take $s \neq 0 \in M$, and let S be the multiplicatively closed set consisting of the powers of s. Now $R_S \subset Q$; on the other hand, the prime ideals of R_S and the prime ideals of R not meeting S are in 1-1 correspondence. Thus R_S is a field, and $R_S = Q$. Therefore, Q is a countably generated R-module; and thus $\mathrm{hd}_R\, Q = 1$ by Theorem 1.3, or the remark following it.

THEOREM 3.3. *Every divisible R-module D is a homomorphic image of an injective R-module; and thus D_T is a direct summand of D.*

PROOF. Since $\mathrm{hd}_R\, Q = 1$ by Lemma 3.2, it follows from Proposition 2.2 that we only need to prove that D_T is a homomorphic image of an injective R-module. By Lemma 3.1 $D_T = \sum_a \oplus D_{T_a}$, where $D_{T_a} = D_T \otimes_R R_{M_a}$ is a divisible R_{M_a}-module. As we have seen in Lemma 3.2, Q is a countably generated R_{M_a}-module; and thus by Theorem 1.3, D_{T_a} is a homomorphic image of an injective R_{M_a}-module. Thus D_{T_a} is a homomorphic image of an injective R-module; and therefore, the same is true of D_T.

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A CHARACTERIZATION OF ALGEBRAIC NUMBER FIELDS WITH CLASS NUMBER TWO[1]

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Let $Z=R(\theta)$ denote an algebraic number field over the rationals with class number h. It is familiar that $h=1$ if and only if unique factorization into prime holds for the integers of Z. For fields with $h\leqq 2$ we have the following criterion.

Theorem. *The algebraic number field Z has class number $\leqq 2$ if and only if for every nonzero integer $\alpha\in Z$ the number of primes π_j in every factorization*

$$\alpha = \pi_1\pi_2 \cdots \pi_k \tag{1}$$

depends only on α.

Suppose first that $h=2$ and consider the factorization into prime ideals

$$(\alpha) = \mathfrak{p}_1 \cdots \mathfrak{p}_s\mathfrak{r}_1 \cdots \mathfrak{r}_t, \tag{2}$$

where the $\mathfrak{p}_j$ are principal ideals while the $\mathfrak{r}_j$ are not. Then

$$\mathfrak{p}_j = (\pi_j) \qquad (j = 1, \cdots, s).$$

Since $h=2$, it follows that

$$\mathfrak{r}_i\mathfrak{r}_j = (\rho_{ij}) \qquad (i, j = 1, \cdots, t);$$

moreover t must be even, $=2u$, say. Thus every factorization into primes implied by (2), for example

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