DIVISIBLE MODULES
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Introduction. Let $R$ be an integral domain with quotient field $Q$, and let $A$ be a module over $R$. $A$ is said to be a divisible $R$-module, if $rA = A$ for every $r \neq 0 \in R$. An element $x \in A$ is said to be a torsion element of $A$, if there exists $r \neq 0 \in R$ such that $rx = 0$. The set of torsion elements of $A$ is a submodule of $A$ called the torsion submodule of $A$, and we will consistently denote it by $A_T$. We will let $E(A)$ denote the injective envelope of $A$ (see [3]); and $\text{hd}_R A$ will denote the homological dimension of $A$ as an $R$-module.

We will study conditions, some necessary, some sufficient, for the torsion submodule of a divisible module to be a direct summand. These will be related to the condition that $\text{hd}_R Q = 1$, where $Q$ is the quotient field of $R$. We will apply these conditions to show that, if $R$ is a Noetherian integral domain in which prime ideals different from zero are maximal, and if $D$ is a divisible module over $R$, then $D$ is a homomorphic image of an injective $R$-module and $D_T$ is a direct summand of $D$. The same conclusions hold, if we merely assume for an arbitrary integral domain that its quotient field is countably generated as a module over the ring.

1. The torsion submodule. It is easy to see that, if $C$ is an injective module over an integral domain $R$, then its torsion submodule $C_T$ is also an injective $R$-module, and therefore a direct summand. Namely, any homomorphism of an ideal of $R$ into $C_T$ can be extended to a homomorphism of $R$ into $C_T$. The following theorem is a generalization of this fact.

Theorem 1.1. Let $R$ be an integral domain and $H$ a homomorphic image of an injective $R$-module. Then $H_T$ is a direct summand of $H$.

Proof. The mapping of a free $R$-module $F$ onto an injective $R$-module $C$ can be extended to a mapping of $E(F)$ onto $C$. We can thus assume that there exists a torsion-free, divisible $R$-module $U$ and an epimorphism $f: U \to H$. Now $H/H_T$, being torsion-free and divisible, is a direct sum of $R$-modules $Q_i$, where $Q_i$ is isomorphic to $Q$ the quotient field of $R$. Let $S_i$ be the inverse image of $Q_i$ under the canonical map $H \to H/H_T$. We will prove that $H_T$ is a direct summand of each $S_i$, and by [2, Lemma 2] this will complete the proof of the theorem.

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Let $y \in S_j - H_T$; then $Ry$ is a torsion-free submodule of $S_j$. Choose $x \in U$ such that $f(x) = y$. Since $U$ is a vector space over $Q$, there exists an $R$-submodule $T_j$ of $U$ such that $T_j \cong Q$ and $x \in T_j$. If $u \neq 0 \in T_j$, there exist $r, s \in R$ such that $ru = sx \neq 0$. Since $rf(u) = sf(x) = sy \neq 0$, we have $f(u) \neq 0$, and thus $f(T_j) \cong T_j \cong Q$.

Since $H = \sum_i S_i$, $f(u) = w + z$, where $w \in \sum_{i \neq j} S_i$ and $z \in S_j$. Hence $rw + rz = rf(u) = sy \in S_j$. Thus $rw \in \sum_{i \neq j} S_i \cap S_j = H_T$. Therefore, $w \in H_T \subset S_j$, and so $f(u) \in S_j$. This shows that $f(T_j) \subset S_j$. Since $f(T_j) \cong Q$, we have $f(T_j) \cap H_T = 0$ and $f(T_j)$ maps onto $Q_j$ under the canonical map $H \to H/H_T$. Thus $S_j = H_T \oplus f(T_j)$; and so $H_T$ is a direct summand of $H$.

**Theorem 1.2.** Let $R$ be an integral domain with quotient field $Q \not\cong R$. Suppose that $D_T$ is a direct summand of $D$ for every divisible $R$-module $D$. Then $h_{dR} Q = 1$.

**Proof.** Let $A$ be any $R$-module and let $E = E(A)$. Since $E$ is an essential extension of $A$, $E/A$ is a torsion $R$-module. Let $G$ be any $R$-module extension of $E/A$ by $Q$. Since both $Q$ and $E/A$ are divisible, $G$ is also divisible. Clearly $E/A$ is the torsion submodule of $G$; and thus by assumption $E/A$ is a direct summand of $G$. Thus $\text{Ext}^1_R(Q, E/A) = 0$ [1, Theorem 14.1.1]. We also have $\text{Ext}^n_R(Q, E) = 0$ for $n > 0$, since $E$ is injective. Therefore, from the exact sequence:

$$\text{Ext}^1_R(Q, E/A) \to \text{Ext}^2_R(Q, A) \to \text{Ext}^2_R(Q, E)$$

we deduce that $\text{Ext}^2_R(Q, A) = 0$. Hence $h_{dR} Q \leq 1$. Since $Q$ is not $R$-projective, $h_{dR} Q = 1$.

Over an arbitrary integral domain it is not true that $D_T$ is a direct summand for every divisible module $D$. For by the above theorem this would imply that $h_{dR} Q = 1$. However, I. Kaplansky has shown (unpublished) that $h_{dR} Q = 1$ for a valuation ring $R$ if and only if $Q$ is a countably generated $R$-module.

**Theorem 1.3.** Let $R$ be an integral domain with quotient field $Q \not\cong R$, and suppose that $Q$ is countably generated as a $R$-module. Then every divisible $R$-module $D$ is a homomorphic image of an injective $R$-module. Thus $D_T$ is a direct summand of $D$, and $h_{dR} Q = 1$.

**Proof.** There exists a countable set of generators $\{q_n\}$ for $Q$ over $R$, and elements $\{a_{n+1}\}$ of $R$ such that $q_1 = 1$ and $a_{n+1}q_{n+1} = q_n$. Let $D$ be a divisible $R$-module, and let $x \neq 0 \in D$. We define a mapping $f$ from the generators $\{q_n\}$ to $D$. Let $f(1) = x$; then there exists $x_1 \in D$ such that $a_2x_2 = x$, and we define $f(q_2) = x_2$. There exists $x_3 \in D$ such that $a_2x_3 = x_2$, and we define $f(q_3) = x_3$. We continue in this way and...
define \( f \) on all the generators \( \{ q_n \} \). It is easily verified that \( f \) induces an \( R \)-homomorphism from \( Q \) into \( D \) such that the image contains \( x \). It is now clear that by taking a big enough direct sum \( G \) of copies of \( Q \) we can define an \( R \)-homomorphism of \( G \) onto \( D \).

It should be remarked that if \( R \) is any integral domain and \( S \) a countable, multiplicatively closed subset of \( R \), then it can be easily shown that if \( F \) is a countably generated free \( R \)-module and \( f \) a suitably chosen mapping of \( F \) onto \( R_S \), then the kernel of \( f \) is free; and thus \( \text{hd}_R R_S \leq 1 \).

2. \( \text{hd}_R Q = 1 \).

**Proposition 2.1.** Let \( R \) be an integral domain with quotient field \( Q \) such that \( \text{hd}_R Q = 1 \). Let \( H \) be an \( R \)-module. Then the following statements are equivalent:

1. \( \text{Ext}_R^1(Q/R, H) = 0 \).
2. Every \( R \)-homomorphism from \( R \) into \( H \) can be extended to an \( R \)-homomorphism from \( Q \) into \( H \).
3. \( H \) is a homomorphic image of an injective \( R \)-module.

**Proof.** That (1) implies (2) follows immediately from the exact sequence:

\[
\text{Hom}_R(Q, H) \rightarrow \text{Hom}_R(R, H) \rightarrow \text{Ext}_R^1(Q/R, H).
\]

That (2) implies (3) is trivial. That (3) implies (1) follows from the fact that \( \text{hd}_R Q/R = 1 \).

**Proposition 2.2.** Let \( R \) be an integral domain with quotient field \( Q \) such that \( \text{hd}_R Q = 1 \). Let \( H \) be an \( R \)-module. Then:

1. If \( H \) is a homomorphic image of an injective \( R \)-module, so is \( H_T \).
2. If \( B \) is a submodule of \( H \) and if \( B \) and \( H/B \) are homomorphic images of injective modules, then so is \( H \).

**Proof.**

1. If \( H \) is a homomorphic image of an injective \( R \)-module, then \( H_T \) is a direct summand of \( H \) by Theorem 1.1. Hence \( H_T \) is also a homomorphic image of an injective \( R \)-module.

2. Suppose that \( B \) and \( H/B \) are homomorphic images of injective \( R \)-modules. We have an exact sequence:

\[
\text{Ext}_R^1(Q/R, B) \rightarrow \text{Ext}_R^1(Q/R, H) \rightarrow \text{Ext}_R^1(Q/R, H/B).
\]

By Proposition 2.1 the two end modules are zero, and thus \( \text{Ext}_R^1(Q/R, H) = 0 \). Hence by Proposition 2.1 again, \( H \) is a homomorphic image of an injective \( R \)-module.
Definition. Let $B$ be a module over an integral domain. Then we will say that $B$ is $h$-reduced, if $B$ has no nonzero submodules which are homomorphic images of injective modules.

Corollary 2.3. Let $A$ be a module over an integral domain $R$ with quotient field $Q$ such that $\text{hd}_R Q = 1$. Then $A$ has a unique largest submodule $H$ which is a homomorphic image of an injective $R$-module, and $A/H$ is $h$-reduced.

Proof. Let $H$ be the sum of all submodules of $A$ which are homomorphic images of injective $R$-modules. It is clear that $H$ is the unique largest submodule of $A$ which is a homomorphic image of an injective $R$-module. Suppose that $B/H$ is a homomorphic image of an injective $R$-module, where $B$ is a submodule of $A$ containing $H$. Then by Proposition 2.2 $B$ is a homomorphic image of an injective $R$-module. Therefore, $B = H$ and $B/H = 0$.

Proposition 2.4. Let $R$ be an integral domain with quotient field $Q$ such that $\text{hd}_R Q = 1$. Let $D$ be a divisible module over $R$, and let $H$ be a submodule of $D_T$ such that $H$ is a homomorphic image of an injective $R$-module. Then $D_T$ is a direct summand of $D$ if and only if $D_T/H$ is a direct summand of $D/H$.

Proof. Suppose that $D_T$ is a direct summand of $D$, and let $S$ be a complementary summand of $D_T$ in $D$. Then $D/H \cong D_T/H \oplus S$, and since $D_T/H$ is the torsion submodule of $D/H$, $D_T/H$ is a direct summand of $D/H$. Conversely, suppose that $D/H = D_T/H \oplus G/H$, where $G$ is a submodule of $D$ containing $H$. Now $G/H$ is torsion-free and divisible, hence injective. Thus by Proposition 2.2 and Theorem 1.1 $H$ is a direct summand of $G$. Let $L$ be a complementary summand of $H$ in $G$. Then it is clear that $D = D_T \oplus L$.

Proposition 2.5. Let $R$ be an integral domain with quotient field $Q$, and let $T$ be an $h$-reduced torsion $R$-module. Then $\text{Ext}^1_R(Q, T) = 0$ if and only if $T \cong \text{Ext}^1_R(Q/R, T)$.

Proof. Since $\text{Hom}_R(Q, T) = 0$, we have an exact sequence:

$$0 \rightarrow \text{Hom}_R(R, T) \rightarrow \text{Ext}^1_R(Q/R, T) \rightarrow \text{Ext}^1_R(Q, T) \rightarrow 0.$$ 

It follows that if $\text{Ext}^1_R(Q, T) = 0$, then $T \cong \text{Ext}^1_R(Q/R, T)$. Conversely, if $T \cong \text{Ext}^1_R(Q/R, T)$, then the above exact sequence shows that $\text{Ext}^1_R(Q, T)$ is a torsion module. However, $\text{Ext}^1_R(Q, T)$ is torsion-free, and thus $\text{Ext}^1_R(Q, T) = 0$.

Corollary 2.6. Let $R$ be an integral domain with quotient field $Q$. 
Then the torsion submodule of a divisible $\mathcal{R}$-module is always a direct summand if and only if the following two conditions hold:

1. $hd_{\mathcal{R}} Q = 1$.
2. $T \cong \text{Ext}_{\mathcal{R}}^1(Q/R, T)$, whenever $T$ is an $h$-reduced, torsion, divisible $\mathcal{R}$-module.

**Proof.** The necessity follows from Theorem 1.2 and Proposition 2.5; the sufficiency follows from Corollary 2.3 and Propositions 2.4 and 2.5.

**Proposition 2.7.** Let $\mathcal{R}$ be an integral domain with quotient field $Q$ such that $hd_{\mathcal{R}} Q = 1$ and gl. dim. $\mathcal{R} \leq 2$. Let $S$ be any torsion-free $\mathcal{R}$-module. Then $hd_{\mathcal{R}} S \leq 1$. Thus if $A$ is an $\mathcal{R}$-module such that $A_T$ is a homomorphic image of an injective $\mathcal{R}$-module, then $A_T$ is a direct summand of $A$.

**Proof.** Let $B$ be any $\mathcal{R}$-module. Then from the exact sequence:

$$0 \rightarrow S \rightarrow Q \otimes_{\mathcal{R}} S \rightarrow Q/R \otimes_{\mathcal{R}} S \rightarrow 0$$

we derive the exact sequence:

$$\text{Ext}_{\mathcal{R}}^2(Q \otimes_{\mathcal{R}} S, B) \rightarrow \text{Ext}_{\mathcal{R}}^2(S, B) \rightarrow \text{Ext}_{\mathcal{R}}^3(Q/R \otimes_{\mathcal{R}} S, B).$$

Since $hd_{\mathcal{R}} Q = 1$ and gl. dim. $\mathcal{R} \leq 2$, the two end modules are zero. Thus $hd_{\mathcal{R}} S \leq 1$, and the rest of the theorem follows immediately.

2. **Krull dimension = 1.**

Throughout this section $\mathcal{R}$ will be a Noetherian integral domain with the property that nonzero prime ideals are maximal. We will let $Q$ be the quotient field of $\mathcal{R}$ and $K = Q/R$.

**Definition.** Let $A$ be an $\mathcal{R}$-module and $M$ a prime ideal of $\mathcal{R}$. We will say that $A$ is $M$-primary, if for any $x \neq 0 \in A$, the order ideal of $x$ is an $M$-primary ideal. If $B$ is any $\mathcal{R}$-module, and $A$ is the set of all elements of $B$ whose order ideal is $M$-primary (together with the element 0), then $A$ is an $M$-primary $\mathcal{R}$-module which we will call the $M$-primary component of $B$.

**Lemma 3.1.** Let $B$ be any torsion $\mathcal{R}$-module. Then $B$ is the direct sum of its $M$-primary components, $M$ ranging over the prime ideals of $\mathcal{R}$. Furthermore, $B \otimes_{\mathcal{R}} R_M$ is the $M$-primary component of $B$.

**Proof.** Let $\{ M_a \}$ be the collection of nonzero prime ideals of $\mathcal{R}$. By [3, Theorem 3.3] $E(B) = \sum_a \oplus E_a$, where $E_a$ is the $M_a$-component of $E(B)$. Let $B_a = B \cap E_a$; then $B_a$ is the $M_a$-component of $B$. Let $x \neq 0 \subseteq B$; then $x = x_1 + \cdots + x_n$, where $x_i \subseteq E_i$. Now $\cap_{i=2}^n M_i \subseteq M_1$.
hence there exists \( s \in \cap_{i} M_i \) such that \( s \in M_1 \). Then there exists an integer \( k > 0 \) such that \( s^k x_i = 0 \) for \( i = 2, \ldots, n \). Hence \( s^k \bar{x} = s^k x_1 \).

There are elements \( m \in M \) and \( t \in R \) such that \( 1 = mt + ts^k \). There is an integer \( q > 0 \) such that \( m^q x_1 = 0 \). Since \( 1 = m^q + rs^k \), \( r \in R \), we have \( x_1 = r s^k x_1 = rs^k x \in B_1 \). Similarly \( x_i \in B_i \) for \( i = 2, \ldots, n \). Thus \( B = \sum \alpha \oplus B_\alpha \).

Let \( M \) be a prime ideal of \( R \). Clearly \( B_\alpha \otimes_R R_{M_\alpha} = 0 \), if \( M_\alpha \neq M_\nu \). Thus \( B \otimes_R R_{M_\nu} = B_\nu \otimes_R R_{M_\nu} \). It is easily seen that the canonical map \( B_\nu \to B_\nu \otimes_R R_{M_\nu} \) is an epimorphism. However, since \( B_\nu \) is \( M_\nu \)-primary, the kernel of this map is zero. Thus \( B_\nu = B_\nu \otimes_R R_{M_\nu} \), and so \( B \otimes_R R_{M_\nu} = B_\nu \).

**Lemma 3.2.** \( \text{hd}_R Q = 1 \).

**Proof.** It is sufficient to prove that \( \text{hd}_R K = 1 \). By Lemma 3.1 \( K = \sum \alpha \oplus K_{M_\alpha} \); thus it is sufficient to prove that \( \text{hd}_R K_{M_\alpha} = 1 \). For this it is sufficient to prove that if \( D \) is any divisible \( R \)-module, then \( \text{Ext}_R(K_{M_\alpha}, D) = 0 \). Let \( A \) be any extension of \( D \) by \( K_{M_\alpha} \); then \( A \) is a divisible \( R \)-module. We have \( D = \sum \alpha \oplus D_{M_\alpha} \) and \( A = \sum \alpha \oplus A_{M_\alpha} \). Clearly \( D_{M_\alpha} = D \cap A_{M_\alpha} \). Hence for \( \alpha \neq \nu \), we have \( D_{M_\alpha} = A_{M_\alpha} \). Thus we have an exact sequence:

\[
0 \to D_{M_\nu} \to A_{M_\nu} \to K_{M_\nu} \to 0,
\]
and all of the modules and mappings of this sequence are \( R_{M_\nu} \)-modules and mappings. Thus we can assume that \( R \) is a local ring with a single nonzero prime ideal \( M \).

Take \( s \neq 0 \in M \), and let \( S \) be the multiplicatively closed set consisting of the powers of \( s \). Now \( R_S \subset Q \); on the other hand, the prime ideals of \( R_S \) and the prime ideals of \( R \) not meeting \( S \) are in 1-1 correspondence. Thus \( R_S \) is a field, and \( R_S = Q \). Therefore, \( Q \) is a countably generated \( R \)-module; and thus \( \text{hd}_R Q = 1 \) by Theorem 1.3, or the remark following it.

**Theorem 3.3.** Every divisible \( R \)-module \( D \) is a homomorphic image of an injective \( R \)-module; and thus \( D_T \) is a direct summand of \( D \).

**Proof.** Since \( \text{hd}_R Q = 1 \) by Lemma 3.2, it follows from Proposition 2.2 that we only need to prove that \( D_T \) is a homomorphic image of an injective \( R \)-module. By Lemma 3.1 \( D_T = \sum \alpha \oplus D_{T_\alpha} \), where \( D_{T_\alpha} = D_T \otimes_R R_{M_\alpha} \) is a divisible \( R_{M_\alpha} \)-module. As we have seen in Lemma 3.2, \( Q \) is a countably generated \( R_{M_\alpha} \)-module; and thus by Theorem 1.3, \( D_{T_\alpha} \) is a homomorphic image of an injective \( R_{M_\alpha} \)-module. Thus \( D_{T_\alpha} \) is a homomorphic image of an injective \( R \)-module; and therefore, the same is true of \( D_T \).
A CHARACTERIZATION OF ALGEBRAIC NUMBER FIELDS WITH CLASS NUMBER TWO

L. CARLITZ

Let $Z = R(\theta)$ denote an algebraic number field over the rationals with class number $h$. It is familiar that $h = 1$ if and only if unique factorization into prime holds for the integers of $Z$. For fields with $h \leq 2$ we have the following criterion.

**Theorem.** The algebraic number field $Z$ has class number $2$ if and only if for every nonzero integer $\alpha \in Z$ the number of primes $\pi_i$ in every factorization

\[ \alpha = \pi_1 \pi_2 \cdots \pi_k \]

depends only on $\alpha$.

Suppose first that $h = 2$ and consider the factorization into prime ideals

\[ (\alpha) = p_1 \cdots p_s r_1 \cdots r_t, \]

where the $p_i$ are principal ideals while the $r_j$ are not. Then

\[ p_i = (\pi_i) \quad (j = 1, \cdots, s). \]

Since $h = 2$, it follows that

\[ r_i r_j = (\rho_{ij}) \quad (i, j = 1, \cdots, t); \]

moreover $t$ must be even, $= 2u$, say. Thus every factorization into primes implied by (2), for example

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