1 - 1 + 1 - 1 + ... = \frac{1}{2}

DONALD J. NEWMAN

It is of course well known that 1 - 1 + 1 - 1 ... is Abel-summable to 1/2, that is to say

\[ 1 - t + t^2 - t^3 + \cdots \to \frac{1}{2} \quad \text{as } t \to 1^- . \]

It can also be shown that

\[ 1 - t + t^4 - \cdots + t^n \cdots \to \frac{1}{2} \quad \text{as } t \to 1^- . \]

If we consider, however, the rate at which these two functions approach their limit (1/2) then we find that these are worlds apart!

In fact

\[
(1 - t + t^2 \cdots) - \frac{1}{2} = \frac{1}{1 + t} - \frac{1}{2} = \frac{1 - t}{2(1 + t)} \sim \frac{1 - t}{4},
\]

whereas an application of the functional equation for the \( \theta \)-function gives

\[
(1 - t + t^4 - \cdots) - \frac{1}{2} \sim \left( \frac{\pi e^{x}}{1 - t} \right)^{1/2} \exp \left( -\frac{\pi^2}{4} \cdot \frac{1}{1 - t} \right).
\]

In this note we show that this anomalous behavior persists for the other functions

\[ f_k(t) = \sum_{n=0}^{\infty} (-1)^n n^k, \quad k = 1, 2, \ldots, \]

that namely \( f_k(t) \) approaches 1/2 very slowly for \( k \) odd and very quickly for \( k \) even.

**Theorem.** If \( f_k(t) \) are defined by (1) then

A. for any fixed odd \( k \) there is a \( c > 0 \) such that \( |f_k(t) - 1/2| > c(1 - t) \),

B. for any fixed even \( k \) there is a \( c > 0 \) such that

\[ |f_k(t) - 1/2| < A \exp (-c/(1 - t)^\alpha), \quad \alpha = 1/(k - 1). \]

**Proof.** Our principal tool will be the Mellin inversion formula which gives

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From this we obtain, by a term by term integration,

\[ \frac{1}{2\pi i} \int_{-2-i\infty}^{2+i\infty} \Gamma(s) x^{-s} ds = e^{-x}, \quad x > 0. \]

Now note that

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{ks}} = \left(1 - \frac{2}{2^{ks}}\right) \sum_{n=1}^{\infty} \frac{1}{n^{ks}} = (1 - 2^{1-ks}) \xi(ks) \]

and obtain

\[ \frac{1}{2\pi i} \int_{-2-i\infty}^{2+i\infty} \Gamma(s) x^{-s} (1 - 2^{1-ks}) \xi(ks) ds = 1 - f_k(e^{-x}). \]

We now shift the contour from the vertical line \( \text{Re } s = 2 \) to the line \( \text{Re } s = -\frac{1}{2k} \). (It is assumed that \( k > 1 \).) The estimates

\[ |\Gamma(s)| < Ae^{-c|t|}, \quad |\xi(ks)| < |t|^A, \quad t = \text{Im } s, A, c > 0, \]

are known to be valid in \(-\frac{1}{k} \leq \text{Re } s \leq 2, |t| > 1 \) and so the Residue theorem is applicable.

However, in \(-\frac{1}{k} \leq \text{Re } s \leq 2, \) the only pole is seen to be at \( s = 0 \) with residue \((1 - 2)\xi(0) = 1/2 \) and so we obtain

\[ \frac{1}{2\pi i} \int_{-\frac{1}{k}-i\infty}^{-\frac{1}{k}+i\infty} \Gamma(s) x^{-s} (1 - 2^{1-ks}) \xi(ks) ds = \frac{1}{2} - f_k(e^{-x}). \]

Let us now utilize the functional equation of the \( \xi \)-function, namely

\[ \xi(s) = \frac{1}{\pi} (2\pi)^s \sin \frac{\pi}{2} s \Gamma(1 - s) \xi(1 - s). \]

If we introduce this into (3) and then call \( 1 - ks = z \) the result is

\[ -\frac{1}{k\pi i} \int_{2-i\infty}^{2+i\infty} \pi^{-s} \Gamma \left( \frac{1 - z}{k} \right) \Gamma(z) \cos \frac{\pi}{2} zx^{(e-1)/k}(1 - 2^{-s}) \xi(z) dz \]

\[ = \frac{1}{2} - f_k(e^{-x}). \]

\textbf{Proof of A.} Here \( k \) is odd. If we shift the contour from 2 to \( 2k \) then the residue theorem is again applicable and it gives one residue at \( z = k+1 \), namely \( cx \) where
\[ c = \frac{2(k)!}{\pi^{k+1}} \left(1 - 2^{-(k+1)}\right) \zeta(k + 1)(-1)^{(k-1)/2}. \]

The remaining contour integral is the same as that in (5) with the limits changed to \(2k - i\infty, 2k + i\infty\). This clearly has magnitude

\[ \leq x^{(2k-1)/k} \int_{2k-i\infty}^{2k+i\infty} M \cdot \left| \Gamma\left(\frac{1 - z}{k}\right) \right| dz = cx^{(2k-1)/k} \]

and so finally, by (5)

\[ \frac{1}{2} - f_k(e^{-z}) = cx + O(x^{2-1/k}), \quad k > 1, \]

which yields A immediately.

**Proof of B.** Here \(k\) is even and so the integrand in (5) is analytic for all \(z\), \(\text{Re} \ z > 0\). We may then shift the contour to the right to the line \(\text{Re} \ z = kM + 1\) (\(M\) = half an odd integer) and no residues will be introduced.

We now estimate the resulting integral. We have

\[ \left| \pi^{-z} x^{(z-1)/k} (1 - 2^{-z}) \zeta(z) \right| < x^M, \]

\[ \left| \Gamma(z) \cos \frac{\pi}{2} \frac{z}{k} \right| \leq (kM)!, \]

\[ \left| \Gamma\left(\frac{1 - z}{k}\right) \right| = \left| \Gamma\left(-M + \frac{iy}{k}\right) \right|, \quad (y = \text{Im} \ z), \]

\[ \leq \left| \Gamma\left(-M + 1 - \frac{iy}{k}\right) \right| \cdot \frac{1}{(M - 1/2)!} \]

\[ = \left(\frac{\pi}{\cosh \frac{y}{k}}\right)^{1/2} \cdot \frac{1}{(M - 1/2)!} \]

and so the resulting integral is in magnitude

\[ \leq c_1 \frac{(kM)!}{(M - 1/2)!} x^M \leq c_2 M^{(k-1)M} x^M. \]

If \(M\) is now chosen close to \(x^{-\alpha}/e\), \(\alpha = 1/k - 1\) then the resulting estimate becomes
and $B$ follows immediately. This completes the proof.

These results seem to point to the fact that of all the series $\sum (-1)^n f(n)$ the one which goes “fastest” to $1/2$ is $\sum (-1)^n n^3$. I wish to thank J. Korevaar for pointing out that this is actually the case, in the following sense:

We have seen that $|\sum (-1)^n f(n) - 1/2| \leq e^{-\phi(t)/(1-t)}$. On the other hand, it is impossible that

$$\left| \sum (-1)^n f(n) - \frac{1}{2} \right| \leq e^{-\phi(t)/(1-t)}$$

where $\phi(t) \geq 0$ is unbounded as $t \to 1$. It is, in fact, a theorem of Korevaar [3], that if

$$|\sum A_n e^{-nu}| \leq w(u), \quad u > 0,$$

and

$$A_n \geq -1, \quad \liminf_{u \to 0^+} u \log w(u) = -\infty,$$

then $\sum A_n e^{-nu} = \text{constant}$.

Changing $u$ into $\log (1/t)$ immediately yields the result stated above.

**Bibliography**