A NOTE ON LACUNARY FOURIER SERIES

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A theorem of Kolmogoroff [2, p. 73] states that if the Fourier series $\mathfrak{S}[f]$ of an L-integrable function $f(x)$ has an infinity of gaps (n_ν, n_ν') for which $n_\nu'/n_\nu \geqq \lambda > 1$, (gaps of Hadamard's type), then $s_{n_\nu} \to f$ for almost all x, and it can be concluded that if $f(x)$ is continuous then this is valid for all x. From this theorem one can derive another theorem of Kolmogoroff: If $f \in L^2$ and $n_{\nu+1}/n_\nu \geqq \lambda > 1$ then $s_{n_\nu} \to f$ for almost all x. Recently R. Gosselin [1] has proved a similar theorem with considerably larger subsequences (n_ν, n_ν'), although with less precision in locating the indices.

In this note, we prove the following theorem for gaps where $n_\nu'/n_\nu \to 1$.

THEOREM. *Let $f(x)$ be continuous at $x = x_0$, and let $\mathfrak{S}[f]$ be a lacunary Fourier series with an infinity of gaps (n_ν, n_ν') for which $n_\nu' - n_\nu \to \infty$, $n_\nu'/n_\nu \to 1$ and*

$$\omega\left(x_0, \frac{\pi}{n_\nu' - n_\nu}\right) \log\left(1 - \frac{n_\nu}{n_\nu'}\right) \to 0, \tag{1}$$

with

$$\omega(x_0, \delta) = \sup_{0 < |t| \leqq \delta} \{ |f(x_0 + t) - f(x_0)| \};$$

then

$$s_{n_\nu}(x_0) \to f(x_0).$$

It is well known [2, p. 45] that $|f(x_0+h) - f(x_0)| = o(\log 1/|h|)^{-1}$ does not ensure the convergence of $\mathfrak{S}[f]$ at the point x_0. From the above theorem it follows that in this case $\mathfrak{S}[f]$ converges if, for example, it is a lacunary Fourier series with $n_\nu = \nu^{1+\epsilon}$, $n_{\nu+1} = (\nu+1)^{1+\epsilon}$ for every $\epsilon > 0$.

PROOF OF THEOREM. Take $n_\nu = n$, $n_\nu' = m$ and denote by $D_n(t)$ and $K_n(t)$ the Dirichlet and Fejér kernels, i.e.

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt, \quad (n+1)K_n(t) = \sum_{\nu=0}^{n} D_\nu(t) = \frac{\sin^2 (n+1)t/2}{2 \sin^2 t/2}.$$

From the identity

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$$mK_{m-1}(t) - nK_{n-1}(t) = \sum_{\nu=n}^{m-1} D_\nu(t)$$

$$= (m-n)D_n(t) + \sum_{\nu=1}^{m-n-1} (m-n-\nu)\cos(n+\nu)t,$$

we have in virtue of the lacunarity of $\mathfrak{S}[f]$

$$s_n - f(x_0) = \frac{1}{\pi}\int_0^\pi \phi(x_0, t)D_n(t)$$

$$= \frac{1}{\pi(m-n)}\int_0^\pi \phi(x_0, t)[mK_{m-1}(t) - nK_{n-1}(t)]dt,$$

with

$$\phi(x_0, t) = f(x_0+t) + f(x_0-t) - 2f(x_0).$$

The last integral can be written in the form

$$\frac{1}{2\pi(m-n)}\left\{\int_0^{\pi/(m-n)} + \int_{\pi/(m-n)}^{\delta} + \int_\delta^\pi\right\}\phi(x_0, t)\frac{\sin^2 m\dfrac{t}{2} - \sin^2 n\dfrac{t}{2}}{\sin^2\dfrac{t}{2}}dt$$

$$= I_1 + I_2 + I_3$$

say. We have for I_1

$$(2)\quad |I_1| \leqq \frac{2}{\pi(m-n)}\,\omega\left(x_0, \frac{\pi}{m-n}\right)\int_0^{\pi/2(m-n)} \frac{|\sin^2 mt - \sin^2 nt|}{\sin^2 t}dt.$$

It remains to estimate

$$I_1' = \int_0^{\pi/2(m-n)} \frac{|\sin^2 mt - \sin^2 nt|}{\sin^2 t}dt.$$

Taking $mt = \tau$, $n/m = \xi$ we obtain

$$(3)\qquad I_1' \leqq \frac{\pi^2}{4}m\int_0^{m\pi/2(m-n)} \frac{|\sin^2\tau - \sin^2\xi\tau|}{\tau^2}d\tau.$$

It follows from

$$\sin^2\tau - \sin^2\xi\tau = \sin(1-\xi)\tau\sin(1+\xi)\tau$$

and from

$$\frac{\sin \alpha t}{t} \leqq \frac{2\alpha}{1+\alpha t},$$

that the integral of the right-hand side of (2) is less than

$$\int_0^\infty \left| \sin (1-\xi)\tau \sin (1+\xi)\tau \right| \frac{d\tau}{\tau^2}$$

$$\leqq 4(1-\xi^2) \int_0^\infty \frac{dt}{(1+(1+\xi)t)(1+(1-\xi)t)}$$

$$= 2(1-\xi^2) \frac{1}{\xi} \log \frac{1+\xi}{1-\xi}.$$

In virtue of (2), (3) and $\xi = n/m$ we have

$$|I_1| \leqq \pi \frac{m+n}{n} \omega\left(x_0, \frac{\pi}{m-n}\right) \log \frac{m+n}{m-n}$$

and the last expression for $m/n \to 1$ is equivalent to the left-hand side of (1).

On the other hand, we obtain

$$|I_2| \leqq \frac{2}{2\pi(m-n)} \pi^2 \omega(x_0, \delta) \frac{m-n}{\pi} = \omega(x_0, \delta),$$

and

$$|I_3| \leqq \frac{C}{(m-n)\delta^2}$$

with an absolute constant C.

Given an ϵ we can choose δ so that $\omega(x_0, \delta) < \epsilon$ and we can suppose $(m-n)$ so large that $I_1 + I_2 + I_3 = o(1)$; this proves the theorem.

References

1. R. P. Gosselin, *On the convergence of Fourier series of functions in an L^p class*, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 392–397.

2. G. H. Hardy and W. W. Rogosinski, *Fourier series*, Cambridge, 1950.

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