S^n, and K is then a fibre bundle over S^n. Consequently we have the following commutative diagram of exact sequences

\[ \cdots \to \Pi_k(O_n) \to \Pi_k(O_{n+1}) \to \Pi_k(S^n) \to \Pi_{k-1}(O_n) \to \Pi_{k-1}(O_{n+1}) \to \cdots \]

\[ \downarrow i_* \quad \downarrow j_* \quad \downarrow i_* \quad \downarrow i_* \]

\[ \cdots \to \Pi_k(K_0) \to \Pi_k(K) \to \Pi_k(S^n) \to \Pi_{k-1}(K_0) \to \Pi_{k-1}(K) \to \cdots \]

where \( j_* \) is the identity. It is easily seen that \( \text{Ker}(i_*: \Pi_k(O_{n+1}) \to \Pi_k(K)) \) is the image under \( \Pi_k(O_n) \to \Pi_k(O_{n+1}) \) of \( \text{Ker}(i_*: \Pi_k(O_n) \to \Pi_k(K_0)) \). Since this last kernel is 0 the theorem follows.

**References**


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**A NOTE ON GAUSS' FIRST PROOF OF THE QUADRATIC RECIPROCITY THEOREM**

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We assume that the reader is familiar with Mathews' exposition [1, pp. 45-50] of the inductive proof of the reciprocity theorem. There are three main cases:

I. \( pRq \),
II. \( pNq, q \equiv 3 \pmod{4} \),
III. \( pNq, q \equiv 1 \pmod{4} \).

In I we have \( e^2 - p = qf \), in II we have \( e^2 + p = qf \). In III we have first the lemma which asserts the existence of a prime \( p' < q \) such that \( qNp' \). This implies \( p'Nq \), so that \( pp'Rq \) and so \( e^2 - pp' = qf \). In each of the cases I and II it is necessary to treat two sub-cases; in case III there are four sub-cases. Thus in all there are eight cases to consider.

We should like to point out in this note that it is possible to handle all cases simultaneously by introducing a little notation. To begin with, we define

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Thus we have the single equation

\[(1) \quad e^2 - rp = qf,\]

where \(e\) is even and \(q, f\) is odd and \(|f| < q\).

Next we put

\[d = (f, rp), \quad f = df', \quad e = de', \quad rp = dd',\]

so that \(1\) becomes

\[(2) \quad de'^2 - d' = qf';\]

moreover

\[(f', dd') = (d, d') = (q, dd') = 1.\]

From \(2\) we get \(qf' \equiv -d' \pmod{4}\), so that

\[(3) \quad q + d' + f' \equiv 1 \pmod{4}.\]

Now from \(2\) we also get

\[\left( \frac{dd'}{f'} \right) = \left( \frac{qdf'}{d'} \right) = \left( \frac{-qd'f'}{d} \right) = 1,\]

which gives

\[\left( \frac{q}{dd'} \right) = \left( \frac{-d'}{d} \right) \left( \frac{d'}{d} \right) \left( \frac{f'}{dd'} \right).\]

We now apply the generalized reciprocity theorem:

\[\left( \frac{m}{n} \right) \left( \frac{n}{m} \right) = -1^{(m-1)(n-1)/4},\]

where \(m\) and \(n\) are odd and relatively prime; also one of the numbers may be negative. The special cases \(m\) or \(n = \pm 1\) are included. Then we get, since \((dd'/f') = 1,\)

\[(4) \quad \left( \frac{q}{dd'} \right) = (-1)^h,\]

where
\[ \lambda = \frac{1}{4} (d - 1)(-d' - 1) + \frac{1}{4} (f' - 1)(dd' - 1). \]

Using (3) we find that
\[ \lambda = \frac{1}{4} (d - 1)(-d' - 1) - \frac{1}{4} (q + d')(dd' - 1) \]
\[ = \frac{1}{4} (d - 1)(-d' - 1) - \frac{1}{4} (d' + 1)(d + d' - 2) \]
\[ - \frac{1}{4} (q - 1)(dd' - 1) \]
\[ = - \frac{1}{4} (d' + 1)(2d + d' - 3) - \frac{1}{4} (q - 1)(dd' - 1) \]
\[ = - \frac{1}{4} (d'^2 - 1) - \frac{1}{2} (d' + 1)(d - 1) - \frac{1}{4} (q - 1)(dd' - 1) \]
\[ = \frac{1}{4} (q - 1)(rp - 1) \pmod{2}; \]

where at the last step we used \( rp = dd' \). Thus (4) becomes
\[ (5) \quad \left( \frac{q}{rp} \right) = (-1)^{(q-1)(rp-1)/4}. \]

In cases I and II (5) is in obvious agreement with the reciprocity theorem; in III there is also agreement since we have \( qNp' \). Thus in III (5) reduces to
\[ \left( \frac{q}{p} \right) = -1, \]

which is the desired relation.

**Reference**


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