A DISCRETE ANALOGUE OF THE WEIERSTRASS TRANSFORM

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1. Introduction and definitions. In this note we consider a discrete analogue of the transform

\[ f(x) = \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{14}} g(y) dy. \]

This transform may be considered as a convolution; accordingly our analogue will take the form

\[ f(n) = \sum_{-\infty}^{\infty} k(n - m) g(m) \]

where the kernel sequence \( k(n) \) is to be appropriately chosen. The choice of \( k(n) \) is motivated as follows.

One way in which the kernel \( e^{-\frac{y^2}{14}} \) arises in the theory of the convolution transform is by consideration of the class of kernels whose bilateral Laplace transform is of the form

\[ \frac{e^{as} e^{bs}}{\prod_{1} \left( 1 - \frac{s}{a_{k}} \right)^{-1} e^{-s/a_{k}}}. \]

When \( c = 0 \) we obtain the class of kernels whose analogue was considered by Pollard and Standish [4] where the kernels formed a subclass of the totally positive sequences. The totally positive sequences can be characterized as sequences having a generating function of the form

\[ e^{as + bs} z^{m} \prod_{1} \frac{(1 + \alpha_{k} z)(1 + \beta_{k} z^{-1})}{(1 - \alpha_{k} z)(1 - \beta_{k} z^{-1})} \]

(see [1]). The factor \( e^{as + bs} z^{m} \) in (1.2) corresponds to the factor \( e^{as} \) in (1.1).

Letting \( a = b = \sqrt{2} \) we have the familiar expansion

\[ e^{(\sqrt{2})(x+s^{-1})} = \sum_{-\infty}^{\infty} I_{m}(v) z^{m} \]

Received by the editors February 10, 1959 and, in revised form, October 22, 1959.

1 This work is part of a doctoral thesis done under the direction of Professor Harry Pollard at Cornell University.

where $I_m(v)$ is the modified Bessel coefficient of the first kind.

We take the sequence $\{I_n(1)\}_{n=-\infty}^{\infty} = \{I_n\}_{n=-\infty}^{\infty}$ as the kernel of our transform

$$f(n) = \sum_{m=-\infty}^{\infty} I_{n-m} g(m).$$

Before proceeding further we list some elementary properties of the functions $I_n(v)$.

(i) $I_n(v) = \sum_{s=0}^{\infty} \frac{(v/2)^{n+2s}}{s!(s+n)!}$, \hspace{1cm} n = 0, \pm 1, \pm \cdots$

(ii) $I_n(v) = I_{-n}(v)$; \hspace{1cm} $I_n(-v) = (-1)^n I_n(v)$,

(iii) $\frac{2(1/2)^{|n|}}{|n|!} \leq I_n(1) \leq \frac{3(1/2)^{|n|}}{|n|!}$,

(iv) $I_n(v + v') = \sum_{m=-\infty}^{\infty} I_{n-m}(v) I_m(v')$.

If we take $g(m) = 2|m| m! |z|! / (1+m^2)$ it is clear, by (iii), that (2.13) converges only for the value $n = 0$. We therefore define the transform (1.3) only when it converges for all $n$.

Define the operators $\delta$ and $\delta^{-1}$ by the equations

$$\delta f(n) = f(n - 1), \hspace{1cm} \delta^{-1} f(n) = f(n + 1).$$

Then by analogy with the theory of the Weierstrass transform we may expect (1.3) to be inverted by the operator $e^{-(1/2)(t+\delta^{-1})}$ interpreted as

$$[e^{-(1/2)(t+\delta^{-1})}] f(n) = \lim_{t \to 0^+} \sum_{n=-\infty}^{\infty} (-1)^n I_m f(n - m) / \Gamma(1 + tm).$$

The method of summation adopted in (1.4) is a regular method introduced by Mittag-Leffler (see [2; p. 72]). The motivation for choosing Mittag-Leffler summation as the "natural" method for the inversion operator is indicated in the table below.

<table>
<thead>
<tr>
<th></th>
<th>Continuous</th>
<th>Discrete</th>
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<tbody>
<tr>
<td>Kernel</td>
<td>$e^{-t^2/4}$</td>
<td>$I_m 1/m!$</td>
</tr>
<tr>
<td>Convergence factor</td>
<td>$e^{-(1/2)\sqrt{t}}$</td>
<td>$1/\Gamma(1 + t</td>
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</table>
Unfortunately the operator (1.4) will not serve to invert all convergent transforms (1.3). For example let
\[ g(m) = \begin{cases} \frac{2^m m!}{m \log m} & m > 1, \\ 0 & m \leq 1. \end{cases} \]
Then, if \( n > 0 \) say, we have
\[ f(n) = \sum_{n}^{\infty} \frac{2^m m!}{(m - n)! n \log m} > \sum_{n}^{\infty} \frac{2^m m!}{m \log m} \]
\[ > \sum_{n}^{\infty} \frac{m^n}{m \log m} \]
If in the last sum above we consider only the term with \( m = [e^{n/2}] \) we obtain
\[ f(n) > O(e^{n^2/4}) \quad n \to + \infty; \]
and (1.4) does not converge for any value of \( t \).
In view of the above we restrict attention to the class of determining sequences \( g(m) \) which satisfy the condition
\[ g(m) = O(|m|^{-x|m|}) \]
as \( |m| \to \infty \), where \( x \) is strictly less than 2. We abbreviate this restriction by writing \( g(m) \in X \).

2. The inversion theorem. We first find the order of the transform.

Theorem 1. If \( g(m) \in X \) then \( f(n) = \sum_{n}^{\infty} I_{n-m} g(m) \) converges absolutely for all \( n \), and
\[ f(n) = O(2^{|n|} |n|! (2/x - 1)^{-|n|}) \quad |n| \to \infty. \]
Proof. The first statement is obvious. To prove the second, note that since the order conditions on \( g(m) \) and \( I_m \) are the same for \( m \to + \infty \) and \( m \to - \infty \) the same is true of \( f(n) \). It is therefore sufficient to prove the theorem for \( n \to + \infty \).
\[ |f(n)| \leq \sum_{n}^{\infty} I_{n-m} |g(m)| + \sum_{n}^{\infty} I_{n-m} |g(m)| = S' + S. \]
Since \( n > 0 \) it is clear that \( S' \leq \text{const.} \sum_{n}^{\infty} (x/2)^n / n! \to 0 \) as \( n \to \infty \).
\[ S \leq \text{const.} 2^n \sum_{n}^{\infty} \frac{(x/2)^m m!}{(m - n)!} = \text{const.} x^n \sum_{n}^{\infty} (m + 1) \cdots (m + n) (x/2)^m. \]
The formula
\[ \sum_{m=0}^{\infty} (m + 1) \cdots (m + n) y^m = n!/(1 - y)^{n+1}, \quad y < 1 \]
may be proved by induction. Hence we have
\[ S \leq \text{const.} \frac{x^n n!}{(1 - x/2)^n} = \text{const.} 2^n n!(2/x - 1)^n. \]

**Theorem 2.** If \( g(n) \in \mathcal{X} \) and \( f(n) = \sum_{m=-\infty}^{\infty} I_{n-m} g(m) \), then
\[ \lim_{t \to 0^+} \sum_{n=-\infty}^{\infty} (-1)^m I_m f(n - m)/\Gamma(1 + t \, |\, m\, |) = g(n). \]

**Proof.** There is no loss of generality in taking \( n=0 \). For any \( t > 0 \)
\[
\sum_{m=-\infty}^{\infty} (-1)^m I_m f(-m)/\Gamma(1 + t \, |\, m\, |)
= \sum_{m=-\infty}^{\infty} (-1)^m I_m/\Gamma(1 + t \, |\, m\, |) \sum_{r=-\infty}^{\infty} I_{-m-r} g(r)
= \sum_{r=-\infty}^{\infty} g(r) \sum_{m=-\infty}^{\infty} (-1)^m I_{m-r} I_m/\Gamma(1 + t \, |\, m\, |) = \sum_{r=-\infty}^{\infty} g(r) \mu(r, t).
\]
The interchange of summations is justified by the estimate of Theorem 1.
Since
\[ \mu(0, 0) = \sum_{m=-\infty}^{\infty} (-1)^m I_{m-1} I_m(1) = \sum_{m=-\infty}^{\infty} I_{m-1} I_m(-1) = I_0(0) = 1 \]
(see iv) the theorem will be established if we show
\[ \lim_{t \to 0^+} \sum_{1}^{\infty} g(r) \mu(r, t) = 0, \]
\[ \lim_{t \to 0^+} \sum_{-1}^{-\infty} g(r) \mu(r, t) = 0. \]

We will prove (2.3); the proof of (2.4) is entirely similar.
As a first step let us replace \( 1/\Gamma(1 + t \, |\, m\, |) \) by the contour integral
\[ \frac{1}{\Gamma(1 + t \, |\, m\, |)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \frac{e^u}{u} \frac{1}{u - t \, |\, m\, |} du, \]
where the path of integration is as shown, the circle being of radius greater than one.
Define the function $K = K(t) = 2^{1/t}(2/x - 1)^{-1/t}$, so that $K \to +\infty$ as $t \to +\infty$. For any $t > 0$ we have

$$2\pi i \sum_{r=1}^{\infty} g(r) \mu(r, t) = \sum_{r=1}^{\infty} g(r) \sum_{m=\infty}^{\infty} (-1)^m I_m I_{m-r} \int_{c_1}^{(0+)} \frac{e^u}{u} u^{-|m|} du$$

$$= \sum_{r=1}^{\infty} g(r) \sum_{m=\infty}^{\infty} (-1)^m (\ldots) \int_{c_1} (\ldots) du$$

$$+ \sum_{r=1}^{\infty} g(r) \sum_{m=\infty}^{\infty} (-1)^m (\ldots) \int_{c_1} (\ldots) du$$

$$+ \sum_{r=1}^{\infty} g(r) \sum_{m=\infty}^{\infty} (-1)^m (\ldots) \int_{c_1} (\ldots) du$$

$$= S + S_1 + S_2.$$

The path $C$ runs from $u = -K(\arg u = -\pi)$ round the circle and back to $u = -K(\arg u = \pi)$, $C_1$ runs from $-\infty$ to $-K(\arg u = -\pi)$, $C_2$ runs from $-K$ to $-\infty(\arg u = \pi)$.

We show now that $\lim_{t \to 0^+} S_1 = 0$; the argument showing $\lim_{t \to 0^+} S_2 = 0$ is the same.

$$|S_1| \leq \sum_{r=1}^{\infty} |g(r)| \sum_{m=\infty}^{\infty} I_m I_{m-r} \int_{c_1}^{(0+)} e^u u^{-|m|} du$$

$$\leq e^{-K} \sum_{r=1}^{\infty} |g(r)| \sum_{m=\infty}^{\infty} I_m I_{m-r} K^{-|m|} = e^{-K} \sum_{m=\infty}^{\infty} I_m K^{-|m|} \sum_{r=1}^{\infty} g(r) I_{m-r}$$

$$\leq e^{-K} \sum_{m=\infty}^{\infty} I_m K^{-|m|} m! |m| \left( \frac{2}{x} - 1 \right)^{-|m|} = e^{-K} \sum_{m=\infty}^{\infty} I_m m! |m| ! = e^{-KB}$$

where $B = \sum_{m=\infty}^{\infty} I_m |m| ! < 9$. Therefore $S_1 \leq e^{-KB} \to 0$ as $t \to 0^+$. 

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It is essential to the proof that $K \to \infty$. This is not the case when $x \leq 2/3$. But in that event the sum (1.4), with $t = 0$, converges, and the proof that (1.4) inverts (1.3) is trivial. We therefore assume from now on that $x > 2/3$. 

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The proof has now been reduced to showing that $S \to 0$ as $t \to 0^+$. To this end we make use of the formula [5, p. 441]

$$I_m(1)I_{r-m}(1) = \frac{2}{\pi} \int_0^{\pi/2} I_r(2 \cos \theta) \cos (2m - r) \theta d\theta.$$

We obtain, setting $u^{-t} = w$,

$$\frac{\pi}{2} S = \sum_{r=1}^{\infty} g(r) \sum_{m=-\infty}^{\infty} (-1)^m \int_0^{\pi/2} I_r(2 \cos \theta) \cos (2m - r) \theta d\theta \int_c \frac{e^u}{u} \frac{dw - |m| du}{w^2}$$

$$= \sum_{r=1}^{\infty} g(r) v(r, t).$$

Estimate of $v(r, t)$. Since $|w| < 1$ along the entire path $C$, we may write

$$v(r, t) = \int_C \frac{e^u}{u} \int_0^{\pi/2} I_r(2 \cos \theta) \left\{ \sum_{m=-\infty}^{\infty} (-1)^m w^m \cos (2m - r) \theta \right\} d\theta du.$$

$$\sum_{m=-\infty}^{\infty} (-1)^m w^{|m|} \cos(2m - r) \theta = \cos(r \theta) \frac{1 - w^2}{1 + 2w \cos 2 \theta + w^2}.$$

We now have

$$v(r, t) = \int_C \frac{e^u}{u} (1 - w^2) \int_0^{\pi/2} \frac{I_r(2 \cos \theta) \cos r \theta}{1 + 2w \cos 2 \theta + w^2} d\theta du.$$

The integral

$$Q_r(w) = \int_0^{\pi/2} \frac{I_r(2 \cos \theta) \cos r \theta}{1 + 2w \cos 2 \theta + w^2} d\theta$$

may be estimated as follows.

Replace $I_r(2 \cos \theta)$ by the sum (i) and interchange the order of summation and integration to obtain

$$\left\{ \sum_{s=0}^{\infty} \frac{1}{s!(r + s)!} \int_0^{\pi/2} \frac{(\cos \theta)^{r+s} \cos r \theta}{1 + 2w \cos 2 \theta + w^2} d\theta \right\}.$$

The integral appearing in (2.5) is now estimated by contour integration. It is easily verified that

$$\int_0^{\pi/2} (\cdots) d\theta = \frac{1}{4} \int_0^{2\pi} (\cdots) d\theta,$$

and so the standard substitution $\cos \theta = 1/2(z + 1/z)$, $z = e^{i\theta}$ yields
\[ i 2^{r+2s+3} \int_0^{\pi/2} (\cdots) d\theta = \int_{|z|=1} \frac{(z^2 + 1)^{r+2s}(z^{2r} + 1)}{z^{2r+2s-1}(wz^4 + (1 + w^2)z^2 + w)} \, dz \]

\[ = \int_{|z|=1} \rho_1(z) \, dz + \int_{|z|=1} \rho_2(z) \, dz \]

where

\[ \rho_1(z) = \frac{(z^2 + 1)^{r+2s}}{z^{2s-1}(wz^4 + (1 + w^2)z^2 + w)}; \]

\[ \rho_2(z) = \frac{(z^2 + 1)^{r+2s}}{z^{2r+2s-1}(wz^4 + (1 + w^2)z^2 + w)}. \]

Both \( \rho_1(z) \) and \( \rho_2(z) \) have poles at \( z=0 \) and at the zeros of \( wz^4 + (1 + w^2)z^2 + w \), that is, at \( z = \pm iw^{1/2}, \pm i/w^{1/2} \). The points \( \pm iw^{1/2} \) lie inside the unit circle, while \( \pm i/w^{1/2} \) lie outside.

**Estimate of \( \int_{|z|=1} \rho_1(z) \, dz \).** Let \( \alpha \) be a real positive number such that

\[ \alpha < \inf_{w \in \mathbb{C}} \frac{1}{2} \left| w^{1/2} \right| = \frac{1}{2} \left( \frac{1}{x} - \frac{1}{2} \right)^{1/2}. \]

\[ \left| \int_{|z|=1} \rho_1(z) \, dz \right| \leq \left| \int_{|z|=\alpha} \rho_1(z) \, dz \right| + \left| \text{Res} \, \rho_1(z) \right|_{z=\pm i w^{1/2}}. \]

\[ \left| \text{Res} \, \rho_1(z) \right|_{z=\pm i w^{1/2}} = \frac{|1 - w|^{r+2s}}{2 |w^4| \left| 1 - w^2 \right|}, \]

\[ \left| \int_{|z|=\alpha} \rho_1(z) \, dz \right| \leq 2\pi \alpha^{2} \frac{(\alpha^2 + 1)^{r+2s}}{\alpha^{2s}} M_1(w) \]

where \( M_1(w) = \sup_{|z|=\alpha} \left| wz^4 + (1 + w^2)z^2 + w \right|^{-1} \). \( M_1(w) \) certainly exists since the choice of \( \alpha \) assures that \( wz^4 + (1 + w^2)z^2 + w \) is bounded away from zero for \( |z| = \alpha \). Now, fix \( \alpha \) so small that \( x(\alpha^2 + 1) < 2 \).

**Estimate of \( \int_{|z|=1} \rho_2(z) \, dz \).** Let \( \beta \) be a real positive number such that

\[ \beta > \sup_{w \in \mathbb{C}} \frac{2}{\left| w^{1/2} \right|} = \frac{2}{\left( \frac{1}{x} - \frac{1}{2} \right)^{1/2}}, \]

\[ \left| \int_{|z|=1} \rho_2(z) \, dz \right| \leq \left| \int_{|z|=\beta} \rho_2(z) \, dz \right| + \left| \text{Res} \, \rho_2(z) \right|_{z=\pm i w^{1/2}}, \]

\[ \left| \text{Res} \, \rho_2(z) \right|_{z=\pm i w^{1/2}} = \frac{|1 - w|^{r+2s}}{2 |w^4| \left| 1 - w^2 \right|}, \]
\[
\left| \int_{z=\beta} \rho_\beta(z) dz \right| \leq 2\pi \beta^2 \frac{(\beta^2 + 1)^{r+2s}}{\beta^{2r+2s}} M_2(w)
\]

where \( M_2(w) = \sup_{|z| < \beta} \left| w z^4 + (1 + w^2) z^2 + w \right|^{-1} \). The choice of \( \beta \) assures that \( M_2(w) \) exists. Now fix \( \beta \) so large that \( x(\beta^2 + 1)/\beta^2 < 2 \).

We are now ready to give the estimate of \( \nu(r, t) \).

\[
| \nu(r, t) | \leq \int_C \left| \frac{e^u}{u} \right| | 1 - w^2 | | Q_r(w) | | du |
\]

where

\[
Q(w) \leq \sum_{s=0}^{\infty} \frac{1}{s!(r+s)!2^{r+2s}} \left\{ \left| 1 - w \right|^{r+2s} + \frac{\pi (\alpha^2 + 1)^{r+2s} M_1}{\alpha^{2s-2}} + \frac{2\pi (\beta^2 + 1)^{r+2s} M_2}{\beta^{2r+2s-2}} \right\} .
\]

The estimate (2.6) enables us to write

\[
\sum_{1}^{\infty} | g(r) | | \nu(r, t) |
\]

\[
= \int_C \left| \frac{e^u}{u} \right| | 1 - w^2 | \left\{ \sum_{1}^{\infty} | g(r) | | Q_r(w) | \right\} du .
\]

The proof is now completed by breaking (2.7) into three parts corresponding to the three sums in (2.6). Call these parts \( T_1, T_2, \) and \( T_3 \). We give the details showing \( T_1 \to 0 \) as \( t \to 0 + \). The details for \( T_2 \) and \( T_3 \) are similar, and somewhat simpler after observing that \( M_1(w) \) and \( M_2(w) \) are bounded for \( u \in C \) (again by the choice of \( \alpha \) and \( \beta \)).

\[
T_1 = \int_C \left| \frac{e^u}{u} \right| | 1 - w^2 | \cdot \left\{ \sum_{1}^{\infty} | g(r) | \sum_{s=0}^{\infty} \frac{1}{s!(r+s)!2^{r+2s}} | 1 - w |^{r+2s} \right\} du .
\]

Since \((r+s)! \geq r!s!\) and since \( 2^{2s}(s!)^2 \geq (2s)! \) we have

\[
\sum_{s=0}^{\infty} \frac{1}{s!(r+s)!2^{r+2s}} | 1 - w |^{r+2s} \leq \frac{1 - w}{1 - w^2} \sum_{s=0}^{\infty} \frac{| 1 - w |^{2s}}{(|w|^{1/2})^{2s}} \frac{1}{(2s)!} \leq \frac{1 - w}{1 - w^2} e^{1 - |w|/|w|^{1/2}},
\]

so that
\[ T_1 \leq \int_C \frac{e^u}{u} e^{1-\frac{1}{2}|u|^{1/2}} \sum_1 \infty g(r) \frac{|1-w|^r}{2^r!} \, du \] 
\[ \leq A_1 \int_C \frac{e^u}{u} e^{1-\frac{1}{2}|u|^{1/2}} \frac{|1-w^{-t}|}{1-x/2 |1-w^{-t}|} \, du \]

where \( A_1 \) is a constant.

Break the path \( C \) into two parts, \( C^* \) and \( C_\ast \) as shown:

Given \( \epsilon > 0 \) choose \( t \) so small that there is a fixed positive number \( N < K \) such that

\[ e^{-N} \int_{C^*} e^{1-\frac{1}{2}|u|^{1/2}} \frac{|1-w^{-t}|}{1-x/2 |1-w^{-t}|} \, du \leq \frac{\epsilon}{2} \]  

and also

\[ \sup_{u \in C^*} |1-w^{-t}| = |1-N^{-t}| \]

\[ \leq \frac{\epsilon}{2} \left( \int_{C^*} \frac{e^u}{u} e^{1-\frac{1}{2}|u|^{1/2}} \frac{1}{1-x/2 |1-w^{-t}|} \, du \right)^{-1}. \]

This can be done by virtue of the fact that \( K \to +\infty \) as \( t \to 0^+ \) and the fact that the integrals in (2.8) and (2.9) are seen, by elementary estimates, to be bounded independently of \( t \). Thus \( T_1 < \epsilon \), and since \( \epsilon \) is arbitrary the assertion follows.

**References**