A THEOREM ON FACTORIZABLE GROUPS

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A group $G$ is said to be factorizable if it contains proper subgroups $H, K$ with the property that $G = HK$. Several results are known which state that $G$ is not simple if $H$ and $K$ satisfy suitable assumptions. The following theorem and its corollaries are of a similar nature but apply only to groups of odd order. The proof is based on a theorem of H. Wielandt (see [2]) which generalized earlier results of W. Burnside and I. Schur.

**Theorem.** Let $G$ be a group of odd order and let $M$ be a maximal subgroup of $G$. Suppose that $A$ is an abelian subgroup of $G$, which has at least one cyclic Sylow subgroup, such that $G = AM$. Then either $G$ has prime order or $G$ contains a proper normal subgroup $N$ which is contained in either $A$ or in $M$.

**Proof.** Assume that no proper normal subgroup of $G$ is contained in $M$. Suppose first that $D = A \cap xMx^{-1} \neq \{1\}$, for some element $x$ in $G$. Since $A$ is abelian and since every subgroup of $G$ conjugate to $M$ is of the form $yMy^{-1}$ for some element $y$ in $A$, it follows that $D$ is contained in every subgroup conjugate to $M$. Hence the intersection of all subgroups conjugate to $M$ is a proper normal subgroup of $G$ which is contained in $M$. This contradicts our assumption. Hence $A \cap xMx^{-1} = \{1\}$ for every element $x$ in $G$.

Let $\pi$ be the permutation representation of $G$ induced by the subgroup $M$. As the kernel of $\pi$ is contained in $M$, it follows from the assumptions that $\pi$ is faithful. As $M$ is a maximal subgroup of $G$, $\pi(G)$ is a primitive group of permutations. Since $A$ intersects no
conjugate of $M$ nontrivially, the restriction of $\pi$ to $A$ is the regular representation of $A$. Since $G$ has odd order, $\pi(G)$ cannot be a doubly transitive permutation group. Hence $\pi(G)$ is a permutation group which satisfies the hypotheses of Wielandt’s theorem (see [2]). Consequently that theorem implies that $A$ has prime order and is normal in $G$. The proof is completed by setting $N = A$.

**Corollary 1.** Let $G$ be a group of odd order which can be represented in the form $G = HA$, where $H$ and $A$ are proper subgroups of $G$, and $A$ is an abelian group with at least one cyclic Sylow subgroup. Then $G$ is not simple.

**Proof.** Let $M$ be a maximal subgroup of $G$ which contains $H$. Then it is easily seen that all the hypotheses of the theorem are satisfied. Thus the result follows from the theorem.

The theorem also yields a result of T. Ikuta (see [1]).

**Corollary 2.** A group of odd order which contains a subgroup of prime index is either cyclic of prime order or is not simple.

**Corollary 3.** Let $G$ be a group of odd order. Suppose that $G = HA$, where $H$ is solvable and $A$ is cyclic, then $G$ is solvable.

**Proof.** The proof is by induction on the order $g$ of $G$. The result is clearly true if $G$ has prime order. Assume now that it has been proved for all groups of order less than $g$. If $G = H$ or $G = A$, the result is trivial. Hence it may be assumed that $H \neq G$, \{1\}. Any subgroup of $G$ which contains $H$ satisfies the same assumptions as $G$. Hence if $M$ is a maximal subgroup of $G$ which contains $H$, it follows from the induction assumption that $M$ is solvable. The theorem now implies that $G$ contains a normal subgroup $N$, such that $N$ is contained in $A$ or in $M$. In either case $N$ is solvable. As $G/N$ satisfies the same assumptions as $G$, the induction hypothesis yields that $G/N$ is solvable. Consequently $G$ is solvable as desired.

The assumption that $G$ has odd order is essential in the theorem and all its corollaries, since the simple group of order 60 satisfies all the other hypotheses of the theorem and its corollaries.

**Bibliography**


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