1. Introduction. Let $D$ be a closed set in $m$-space, and let $V$ be a linear space of real valued continuous functions defined on $D$. For any $f \in V$, let $Z(f)$ be its zero-set, the set of $p \in D$ with $f(p) = 0$. If $m = 1$, and $V$ is a class of well behaved functions, then $Z(f)$ will be discrete except for $f$ identically zero. This behavior depends strongly upon $m$. When $m \geq 2$, then—except in special cases—one expects $Z(f)$ to be a nowhere dense set which is locally of dimension $m - 1$.

In this note, we shall be concerned with certain distribution-properties of the zero-sets for functions of $m$ variables which do not seem to have been explicitly stated before; as will be seen, they do not depend upon the nature of the functions in $V$ but only upon the dimension of $V$. If $\dim (V) = N$, then there exists a system of $N$ nonempty disjoint relatively open sets $D_1, D_2, \ldots, D_N$ in $D$ such that if $f$ and $g$ are in $V$, and if they agree at some point in each of the sets $D_j$, then $f = g$. Otherwise stated, if $f \in V$ and $Z(f)$ intersects each of the sets $D_j$, then $f \equiv 0$.

If $D$ is the plane, and $V$ the class of functions $f(x, y) = ax + by + c$, then the assertion is merely that there are three open sets $D_j$ which cannot all be cut by a single straight line. Again, choose $V$ as the space of polynomials in $x, y, z$ of degree at most 2. If $f \in V$ but $f \not\equiv 0$ then $Z(f)$ is a quadric surface in 3-space. Any set of nine points in 3-space lies in some set $Z(f)$; most sets of ten points do not. If we choose a set of points $p_1, \ldots, p_{10}$ through which no quadric passes, and then surround each by a sufficiently small open sphere, we obtain ten open sets $D_1, \ldots, D_{10}$ which form a system of uniqueness domains for the space $V$.

2. Uniqueness domains. Let $V$ be any subspace of $C[D]$ of dimension $N$, and let $\phi_1, \phi_2, \ldots, \phi_N$ be a basis for $V$.

**Theorem 1.** There exist disjoint nonempty sets $D_1, D_2, \ldots, D_N$, relatively open in $D$, and a constant $B$ such that for any $f \in V$, and $p \in D$, and any choice of $p_i \in \overline{D}_i$

$$|f(p)| \leq B \{ |f(p_1)| + |f(p_2)| + \cdots + |f(p_N)| \}.$$  

**Proof.** Let $S$ be the set of all points in $N$-space of the form $x = \langle \phi_1(p), \phi_2(p), \ldots, \phi_N(p) \rangle$ for $p \in D$. If $S$ were to lie in a proper...
subspace of $N$-space, there would exist constants $b_1, b_2, \cdots, b_N$ such that $\sum b_j x_j = 0$ for all $x \in S$. This would imply that $\sum b_j \phi_j = 0$ so that the functions $\phi_j$ would not have been independent. We can therefore assume that $S$ generates all of $N$-space. There then exist points $p_1, p_2, \cdots, p_N$ in $D$ such that the $N$ points $(\phi_1(p_1), \phi_2(p_1), \cdots, \phi_N(p_1))$, for $i = 1, 2, \cdots, N$, are linearly independent. Accordingly, the $N$ by $N$ matrix

$$
\Delta = \begin{pmatrix}
\phi_1(p_1) & \phi_2(p_1) & \cdots & \phi_N(p_1) \\
\phi_1(p_2) & \phi_2(p_2) & \cdots & \phi_N(p_2) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_1(p_N) & \phi_2(p_N) & \cdots & \phi_N(p_N)
\end{pmatrix}
$$

is nonsingular.

For any point $p \in D$, let $L_p$ be the functional defined on $C[D]$ by $L_p(f) = f(p)$. The functionals $L_{p_i}$, for $i = 1, 2, \cdots, N$, are linearly independent; for, if $\sum c_i L_{p_i} = 0$, then for each $j = 1, 2, \cdots, N$, we would have

$$0 = \sum c_i L_{p_i}(\phi_j) = \sum c_i \phi_j(p_i)$$

and by the nonsingularity of $\Delta$, $c_1 = c_2 = \cdots = 0$. Since $\dim(V) = N$, its dual space $V'$ is also of dimension $N$. Accordingly, $L_{p_1}, L_{p_2}, \cdots, L_{p_N}$ constitutes a basis for $V'$. In particular then, if $f \in V$ and $L_{p_i}(f) = f(p_i) = 0$ for $i = 1, 2, \cdots, N$, then $L(f) = 0$ for every $L \in V'$, and $f = 0$. This argument rests upon the nonsingularity of $\Delta$; since the functions $\phi_1, \phi_2, \cdots, \phi_N$ are continuous on $D$, $d(p_1, p_2, \cdots, p_N) = \det(\Delta)$ is a continuous function on $D^N = D \times D \times \cdots \times D$. In particular, we can choose nonempty open sets $D_1, D_2, \cdots, D_N$ in $D$ so that $d(p_1, p_2, \cdots, p_N) \neq 0$ for $p_i \in D_i$. As before, the functionals $L_{p_1}, L_{p_2}, \cdots, L_{p_N}$ will form a basis for $V'$ for any choices of the points $p_i$, with $p_1 \in D_1, p_2 \in D_2, \cdots, p_N \in D_N$. Hence, if $f \in V$, and $Z(f)$ touches each of the sets $D_i$, $i = 1, 2, \cdots, N$, then $f = 0$. This is part of the statement (1). To obtain the more complete formulation, we observe that the sets $D_i$ can be taken so that $\inf |d(p_1, \cdots, p_N)| = \delta > 0$, for all $p_i \in \overline{D_i}$, $i = 1, 2, \cdots, N$. Let $P = (p_1, p_2, \cdots, p_N)$, and let $C_j(r, P)$ be the unique numbers defined by

$$\sum_{j=1}^{N} C_j(r, P) \phi_j(p_k) = \begin{cases} 1 & \text{if } k = r, \\ 0 & \text{if } k \neq r \end{cases}$$

for $r = 1, 2, \cdots, N$, and any $P$ in $D_1 \times D_2 \times \cdots \times D_N$. For each $j$ and $r$, $C_j(r, P)$ is continuous in $P$, and since $|d(P)| \geq \delta$, there exists a constant $B_0$, determined by the functions $\{\phi_j\}$ and the sets $D_i$, such that
Setting \( \psi_r = \sum_{j=1}^{N} \phi_j \) for any \( f \in V \),

\[
\sum_{r=1}^{N} f(p_r) \psi_r.
\]

Moreover, for any \( r \),

\[
\|\psi_r\| \leq \sum_{j=1}^{N} |C_j(r, P)| \|\phi_j\| \leq B_0 \sum_{j=1}^{N} \|\phi_j\| \leq B
\]

where we have used \( \|g\| \) to denote \( \sup_{g \in D} |g(g)| \), for any \( g \in C[D] \). Note that \( B \) again is determined only by the \( \{\phi_j\} \) and the \( D_i \). Returning to (2), we see that

\[
\|f\| \leq B \sum_{r=1}^{N} |f(p_r)|
\]

holding for any choice of \( p_i \in D_i \), for \( i = 1, 2, \ldots, N \).

The conclusion of Theorem 1 can be restated in an interesting way. Introduce a special functional \( M \) on \( C[D] \):

\[
M(f) = \sum_{1 \leq i \leq N} \min_{p_i \in \overline{D_i}} |f(p_i)|.
\]

If we were to replace \( \text{"min"} \) by \( \text{"max"} \), this would define a seminorm on \( C[D] \). On the finite dimensional subspace \( V \), it would in fact be a norm, and would therefore have to define there the unique locally convex linear topology. As defined above, however, \( M \) is not a seminorm, for it is not true in general that \( M(f+g) \leq M(f) + M(g) \). However, Theorem 1 implies that \( M \) still defines a Hausdorff topology on \( V \); if \( M(f) = 0 \), then \( f \) must take the value zero somewhere in each of the sets \( \overline{D_i} \), and \( f \equiv 0 \). The essence of the more general formulation (1) is that this topology again coincides with the unique linear topology in \( V \), and there must exist a constant \( B \) such that \( \|f\| \leq BM(f) \), for all \( f \in V \).

Another familiar instance of this general principle is the fact that in a finite dimensional function space, uniform convergence and \( L^1 \) (or \( L^p \)) convergence coincide. This is usually formulated as an inequality: there is a constant \( A \) such that

\[
|f(p)| \leq A \int_D |f|
\]
for all \( p \in D_i \), and \( f \in V \). This follows at once from (1). Let \( \delta \) be the smallest of the measures \( \mu(D_i) \). Then,

\[
\int_D |f| \geq \sum_{i=1}^{N} \int_{D_i} |f| \geq \sum_{i=1}^{N} \mu(D_i) \min_{p_i \in D_i} |f(p_i)| \geq \delta M(f) \geq \frac{\delta}{B} \|f\|.
\]

3. Generalizations. It is natural to ask if there are any other necessary restrictions upon the sets \( \{D_i\} \) if they are to be uniqueness domains. This is answered by the next statement.

**Theorem 2.** If \( D_1, D_2, \ldots, D_N \) is any collection of open disjoint nonempty subsets of \( D \), then there is a subspace \( V \) of dimension \( N \) in \( C[D] \) such that if \( f \in V \), and if \( Z(f) \) touches each of the sets \( D_i \), then \( f = 0 \).

Choose functions \( \phi_i \) so that \( \phi_i(p) \) is the distance from \( p \) to the closed complement of \( D_i \). Then, \( \phi_i \) vanishes off \( D_i \), but is strictly positive on \( D_i \). Let \( V \) be the set of functions \( f = \sum c_i \phi_i \). If \( f \) takes the value 0 at some point of \( D_i \), then it follows that \( c_i = 0 \); accordingly, if \( Z(f) \) contains at least one point of each \( D_i \), then \( f = 0 \).

From this, by a method suggested by the referee, other examples can be constructed. Let \( \theta \) be any continuous mapping of \( D \) onto another set \( D' \), and let \( V' \) be an \( N \) dimensional subspace of \( C[D'] \). Suppose that \( \{D'_1, \ldots, D'_N\} \) is a collection of uniqueness domains in \( D' \), and let \( D_i = \theta^{-1}(D'_i) \). For any \( \phi' \in V' \), define a function \( \phi \) on \( D \) by \( \phi(p) = \phi'(\theta p) \). Then, this yields a linear space \( V \subset C[D] \) having \( \{D_1, \ldots, D_N\} \) for uniqueness domains. In particular, the sets \( D_i \) can be selected so that their union exhausts all but a nowhere dense subset of \( D \), and this can be achieved with many different choices of \( V \).

This is no longer possible in general if we wish to retain the more precise inequality (1) given in Theorem 1. Suppose that we have a space \( V \) of dimension \( N \), and a collection of sets \( \{D_i\} \) such that for any \( f \in V \),

\[
\|f\| \leq B \sum_{i=1}^{N} \inf_{p_i \in D_i} |f(p_i)|.
\]

Suppose that there is a point \( q \) which is a common boundary point for all the sets \( D_i \). Then \( N = 1 \). For, if \( f \in V \) and \( f(q) = 0 \), then by the inequality and the continuity of \( f \), \( f \equiv 0 \). Suppose \( \phi_1 \) and \( \phi_2 \) are two independent functions in \( V \); set \( f = \phi_2(q)\phi_1 - \phi_1(q)\phi_2 \) and have \( f(q) = 0 \).