1. Let $E$ be a locally compact Hausdorff space, $\mu$ a Radon measure, [3], on $E$, and $\Omega(E, \mu)$ the space of equivalence classes of locally integrable functions on $E$ with respect to the measure $\mu$. Here, two functions $f$ and $g$ are equivalent if $f - g = 0$ except on a set $S$ which meets every compact set in a set of measure zero; the notation $f = \lim f_n$ applies to equivalence classes. For a subset $\Gamma$ of $\Omega$, let

$$\Lambda(\Gamma) = \left\{ f \in \Omega : \int f g d\mu < \infty \text{ for all } g \in \Gamma \right\}.$$ 

The sets $\Lambda = \Lambda(\Gamma)$ and $\Lambda^* = \Lambda(\Lambda)$ are vector lattices, [1], and are called associated Köthe spaces. Initially Köthe and Toeplitz, [9], and later Köthe, in a series of papers of which [10] is representative, studied these spaces for the case where $E$ is the space of natural numbers with the discrete topology and $\mu(n) = 1$ for every natural number $n$. Dieudonné, [4], extended the theory to the case for which $E$ is $\sigma$-compact. Köthe spaces, which are also Banach spaces, were studied by Lorentz and Wertheim [7] for the case where $E = [0, 1]$ and $\mu$ is Lebesgue measure. Examples of Köthe spaces are the Lebesgue spaces $L^p$, the Orlicz spaces $L^\Phi$, and arbitrary intersections of such spaces.

For each Köthe space $\Lambda$, the associated space $\Lambda^*$ determines a family of topologies in $\Lambda$. These topologies are locally convex, Hausdorff, and are compatible with the order relation in $\Lambda$. Among them there is a weakest and a strongest. It should be observed that the strongest of these topologies may be strictly stronger than the Mackey topology $m(\Lambda, \Lambda^*)$, [2]. In the case that $E$ is $\sigma$-compact Dieudonné [4] showed that the Köthe space $\Lambda$ is complete for each of these topologies. Later, Goffman [5], using the work of Nakano, observed that the restriction of $\sigma$-compactness is not necessary.

In this paper a characterization is obtained of those Köthe spaces which, with their strongest Köthe topologies $S(\Lambda, \Lambda^*)$, are Banach spaces. A slight modification of these conditions gives a characterization of those Köthe spaces $\Lambda$, which, with their strongest Köthe topologies $S(\Lambda, \Lambda^*)$, are Fréchet spaces. It is shown that $\Lambda$ with the
topology \( S(\Lambda, \Lambda^*) \) is a Banach space if and only if \( \Lambda^* \) is a Banach space with the topology \( S(\Lambda^*, \Lambda) \). If \( \Lambda \) with the topology \( S(\Lambda, \Lambda^*) \) is a Fréchet space, but not a Banach space, then \( \Lambda^* \) is not metrizable for the topology \( S(\Lambda^*, \Lambda) \). As \( \Lambda^* \) is not in general the topological dual of \( \Lambda \) with the topology \( S(\Lambda, \Lambda^*) \) these results do not follow from the standard theory. In general the Köthe space \( \Lambda \) has more than one Köthe topology. In §5 is considered a case where this is not so.

2. Let \( W \) be a family of weakly bounded subsets of \( \Lambda^* \) whose union is all of \( \Lambda^* \). The weak duality determined for \( \Lambda \) and \( \Lambda^* \) by the bilinear form

\[
(f, g) = \int fg d\mu,
\]

\( f \in \Lambda, \ g \in \Lambda^* \) makes it possible to define for \( \Lambda \) the topology of uniform convergence on sets in \( W \). With this topology \( \Lambda \) is a locally convex space. If the sets in \( W \) are normal\(^2\) as well as weakly bounded then this topology is called a Köthe topology and is denoted by \( \kappa_w(\Lambda, \Lambda^*) \). The strongest and weakest such topologies are determined when \( W \) contains all weakly bounded normal subsets of \( \Lambda^* \) and \( W \) contains only the normal closures\(^3\) of points in \( \Lambda^* \) respectively. The weak topology \( \sigma(\Lambda, \Lambda^*) \) is in general weaker than the weakest Köthe topology.

3. For a nonempty class \( C \) of positive integrable functions \( c(t) \) on \([0, 1]\), Lorentz and Wertheim [7], define the Köthe space \( X(C) \) as the set of measurable functions \( f(t) \) for which

\[
\|f\| = \sup_{c \in C} \int_0^1 f(t)c(t) dt < \infty.
\]

They show that if certain additional assumptions are placed on the set \( C \), then their definition is equivalent to that given by Dieudonné, and their space \( X(C) \) is a Banach lattice. Their additional conditions give the motivation for the following definition.

A nonempty subset \( A \) of the Köthe space \( \Lambda \) is said to satisfy condition \((*)\), if for each nondecreasing sequence \((f_n)\) of nonnegative functions in \( A \), there exists a function \( f \) in \( A \) such that \( f \geq f_n \) for \( n = 1, 2, \ldots \).

\(^2\) A subset \( A \) of partially ordered vector space \( X \) is normal if \( x \in X, y \in A \), and \( |x| \leq |y| \) implies \( x \in A \).

\(^3\) The \( P \) closure, of a subset \( A \) of \( \Lambda \), is the smallest subset of \( \Lambda \) containing \( A \) which is \( P \).
Normal convex subsets $A$ of $\Lambda$ which satisfy condition (*) will occur so often in the discussion that they will be referred to as admissible sets. Admissible sets play a fundamental role for:

**Theorem 1.** Every admissible subset of the Köthe space $\Lambda$ is weakly bounded.

**Proof.** Suppose that $A$ is an admissible subset of $\Lambda$ which is not weakly bounded. There exists a function $g$ in $\Lambda^*$, and a sequence $(f_n)$ in $A$ such that

$$\left| \int f_n g d\mu \right| > 2^n, \quad n = 1, 2, \ldots$$

Let $h_n = \sum_{i=1}^{n} 2^{-i} |f_i|$. Because $h_{n+1} \equiv h_n \geq 0$ for each integer $n$, and because $A$ is admissible there exists a function $h$ in $A$ such that $h \equiv h_n$ for $n = 1, 2, \ldots$. Now

$$\int |hg| \, d\mu \geq \int |h_n g| \, d\mu > n, \quad n = 1, 2, \ldots$$

This implies that $|hg|$ is not integrable, while $hg$ is integrable. Hence $A$ is weakly bounded.

For a subset $A$ of the Köthe space $\Lambda$,

$$A^0 = \left\{ g \in \Lambda^* : \left| \int fg d\mu \right| \leq 1 \text{ for all } f \in A \right\}.$$ 

If $A$ is an absorbing subset of $\Lambda$, then $A^0$ is a weakly bounded subset of $\Lambda^*$. If $A$ is a weakly bounded subset of $\Lambda$, then $A^0$ is an absorbing subset of $\Lambda^*$. For proofs of these facts together with the fact that $(A^0)^0 = A^{00}$ is the weak convex closure of $A$, see [2].

A consequence of the following lemma is that every Köthe topology $\mathcal{K}_\omega(\Lambda, \Lambda^*)$ is compatible [14] with the natural order relation in $\Lambda$.

**Lemma 1.** If $A$ is a normal subset of the Köthe space $\Lambda$, then $A^0$ is a normal subset of $\Lambda^*$.

**Proof.** Observe first that

$$\sup_{f \in A} \left| \int fg d\mu \right| = \sup_{f \in A} \int |fg| \, d\mu$$

for any $g$ in $A^0$, that is, if $g$ is in $A^0$, then $|g|$ is in $A^0$. If $h$ is in $\Omega$ and $|h| \leq |g|$ for some $g$ in $A^0$, then
\[ 1 \geq \sup_{f \in A} \int |fg| \, d\mu \geq \sup_{f \in A} \int |fh| \, d\mu. \]

Hence \( h \) is in \( A^0 \). It follows that \( A^0 \) is normal.

**Theorem 2.** Suppose that \( A \) is a weakly bounded subset of the Köthe space \( \Lambda \). If \( \bar{A} \) is the normal convex closure of \( A \), then \((\bar{A})^{00}\) satisfies condition (*)

**Proof.** The set \((\bar{A})^{00}\) is by [4] weakly bounded and by Lemma 1 normal. From previous remarks \((\bar{A})^{00}\) is also convex and weakly closed.

Suppose that \((f_n)\) is any nondecreasing sequence of nonnegative functions in \((\bar{A})^{00}\). For any function \( g \) in \( \Lambda^* \) there exists a positive constant \( M(g) \) such that

\[ \int f_n |g| \, d\mu \leq M(g), \quad n = 1, 2, \ldots. \]

Let \( f = \lim f_n \) so that \( f|g| = \lim f_n|g| \). By Fatou's lemma

\[ \int f |g| \, d\mu \leq \lim \inf \int f_n |g| \, d\mu \leq M(g). \]

Since the characteristic functions of compact subsets of \( E \) are in the associated space \( \Lambda^* \), the function \( f \) is locally integrable, and hence is in \( \Omega(E, \mu) \). As the function \( g \) was chosen arbitrarily from \( \Lambda^* \), (1) shows that \( f \) is in \( \Lambda \). It remains to show that \( f \) is a weak limit of the sequence \((f_n)\), and hence is in \((\bar{A})^{00}\).

To this end, suppose that a positive number \( \epsilon \) and a function \( g \) from \( \Lambda^* \) have been chosen arbitrarily. Because \( f|g| \) is integrable, there exists a compact subset \( K \) of \( E \), such that

\[ \int_{c(K)} f |g| \, d\mu \leq \frac{\epsilon}{6}. \]

A positive number \( \delta \) exists such that for every measurable subset \( F \) of \( K \) for which \( \mu(F) < \delta \),

\[ \int_{F} f |g| \, d\mu < \frac{\epsilon}{6}. \]

By Egoroff's theorem, a measurable subset \( F \) of \( K \) exists, such that \( \mu(F) < \delta \), and \( f_n \) converges uniformly to \( f \) on \( K \sim F \). Choose a number \( N \) such that \( n \geq N \implies |fg - f_ng| < \frac{\epsilon}{3\mu(K \sim F)} \). It follows that
As $\epsilon$ and $g$ were chosen arbitrarily,

$$\lim_{n \to \infty} \left| \int (f - f_n)g d\mu \right| = 0$$

for every $g$ in $\Lambda^*$. Hence $f$ is the weak limit of the sequence $(f_n)$.

**Corollary 1.** A normal subset $A$ of the Köthe space $\Lambda$ is weakly bounded if and only if $(A)^0$ satisfies condition (*).

One obtains from this corollary that $\mathcal{K}_w(\Lambda, \Lambda^*) = S(\Lambda, \Lambda^*)$, if $W$ contains all admissible subsets of $\Lambda^*$.

4. It is known: if $X$ and $Y$ are two linear spaces in duality, then every weakly bounded subset of $X$ is bounded for the Mackey topology $m(X, Y)$, [2]. For a Köthe space $\Lambda$ it was shown by Dieudonné in [4] that the bounded subsets for the weak topology $\sigma(\Lambda, \Lambda^*)$ are the same as the bounded subsets for the strong topology. His proof is based on results of Mackey [12; 13]. A simple direct proof will now be given.

**Lemma 2.** Every weakly bounded subset of the Köthe space $\Lambda$ is bounded for the strong topology $S(\Lambda, \Lambda^*)$.

**Proof.** Suppose the weakly bounded subset $A$ of $\Lambda$ is not bounded for the topology $S(\Lambda, \Lambda^*)$. By Corollary 1 an admissible subset $B$ of $\Lambda^*$ and a sequence $(f_n)$ in $A$ exist such that for each integer $n$ a function $g_n$ in $B$ can be found for which

$$\int |g_n f_n| d\mu \leq n 2^n.$$ 

Set $h_n = \sum_{k=1}^{n} 2^{-k} |g_k|$ for $n = 1, 2, \cdots$. As $(h_n)$ is a nondecreasing sequence of nonnegative functions in $B$, there exists a function $h$ in $B$ such that $h \geq h_n$ for $n = 1, 2, \cdots$. The normal closure $\overline{A}$ of $A$ is weakly bounded [4], but

$$\sup_{f \in \overline{A}} \left| \int f h d\mu \right| \geq \int \left| f_n h \right| d\mu \geq n$$

for $n = 1, 2, \cdots$ implies that $\overline{A}$ is not weakly bounded. Hence the assumption that $A$ is not bounded for the topology $S(\Lambda, \Lambda^*)$ is false.
Lemma 3. If $A$ is an absorbing admissible subset of the Köthe space $\Lambda$, then $A^0$ is a bounded neighborhood of zero for the strong topology.

Proof. Because $A$ is normal and absorbing $A^0$ is normal and weakly bounded. It follows that $A^0$ is a neighborhood of zero for the topology $S(\Lambda, \Lambda^*)$.

Since $A$ is admissible, it is by Theorem 1 weakly bounded. Therefore, $A^0$ is absorbing. But $A^0$ being absorbing implies that $A^{00}$ is weakly bounded. It follows from Lemma 2 that $A^{00}$ is bounded for the strong topology.

Theorem 3. The Köthe space $\Lambda$ is a Banach space for the strong topology if and only if $\Lambda$ contains an absorbing admissible set.

Proof. Suppose that $A$ is an absorbing admissible subset of $\Lambda$, then $A^{00}$ is a convex bounded neighborhood of zero for the strong topology $S(\Lambda, \Lambda^*)$. By a theorem of Kolmogoroff [9] this topology is equivalent to a norm topology. Since $\Lambda$ is complete for $S(\Lambda, \Lambda^*)$ [5], it is a Banach space.

Conversely if $\Lambda$ is a Banach space for $S(\Lambda, \Lambda^*)$ then the unit sphere $A$ in $\Lambda$ is a weakly bounded absorbing set. It follows from Theorem 2 that $(A)^{00}$ is an absorbing admissible subset of $\Lambda$.

Corollary 2. The Köthe space $\Lambda$ is a Banach space for the topology $S(\Lambda, \Lambda^*)$ if and only if $\Lambda^*$ is a Banach space for the topology $S(\Lambda^*, \Lambda)$.

Proof. Suppose that $\Lambda$ is a Banach space for the topology $S(\Lambda, \Lambda^*)$ and let $A$ be an absorbing admissible subset of $\Lambda$. By Lemma 1, $A^0$ is normal. As $A$ is weakly bounded and absorbing, $A^0$ is absorbing and weakly bounded. It is clear that $A^0$ is convex; therefore, $A^0$ is an admissible absorbing subset of $\Lambda^*$. Hence $\Lambda^*$ is a Banach space.

The converse is proved by interchanging the roles of $\Lambda$ and $\Lambda^*$ in the preceding argument.

Corollary 2 shows that the statement of Theorem 3 could be made in terms of the subsets of $\Lambda^*$. With this alternate approach in mind one states:

Theorem 4. The Köthe space $\Lambda$ is a Fréchet space for the strong topology if and only if $\Lambda^*$ contains a sequence $B_1 \subset B_2 \subset \cdots$ of admissible subsets, such that every weakly bounded subset $B$ of $\Lambda^*$ is contained in one of the $B_k$.

Proof. Suppose $\Lambda^*$ contains such a sequence $B_1 \subset B_2 \subset \cdots$ of admissible sets. The topology of uniform convergence on these sets is clearly the strong topology for $\Lambda$. As this topology has a countable
for the neighborhood system of zero, it is a metric topology. The completeness of \( \Lambda \) with this topology follows from [5].

Conversely, suppose that \( \Lambda \) is a Fréchet space for the strong topology. A sequence \( A_1 \supset A_2 \supset \cdots \) of normal convex subsets of \( \Lambda \) can be found, which is a base in this topology for the neighborhood system of zero. The sets in the sequence \( A_1^0 \subset A_2^0 \subset \cdots \) are admissible, for, each \( A_i \) is both normal and absorbing. This sequence satisfies the further condition that each weakly bounded subset \( B \) of \( \Lambda^* \) is contained in one of the \( A_i^0 \). For, if \( B \) is a weakly bounded subset of \( \Lambda^* \), and \( \tilde{B} \) is its normal closure, then \( \tilde{B}^0 \) is a neighborhood of zero in \( \Lambda \). This implies an integer \( k \) can be found such that \( A_k \subset \tilde{B}^0 \). Hence \( B \subset A_k^0 \).

**Remark.** If \( \Lambda \) is a Fréchet space with the strong topology, Corollary 2 suggests asking, is \( \Lambda^* \) a Fréchet space with its strong topology. The following discussion shows that if \( \Lambda \) is a Fréchet space, but not a Banach space, then \( S(\Lambda^*, \Lambda) \) is not a metric topology. Suppose \( \Lambda^* \) with \( S(\Lambda^*, \Lambda) \) is a Fréchet space even though \( \Lambda \) with \( S(\Lambda, \Lambda^*) \) is not a Banach space. By Theorem 4 a sequence \( B_1 \subset B_2 \subset \cdots \) of admissible subsets of \( \Lambda^* \) can be found whose union is \( \Lambda^* \). The sets \( B_1^{00}, B_2^{00}, \cdots \) are again admissible, and in addition are strongly closed. As the union of the sets \( B_i^{00} \) is again \( \Lambda^* \), the Baire category theorem ensures the existence of an integer \( k \) such that \( B_k^{00} \) has an interior point. Thus \( B_k^{00} \) is an absorbing admissible set, that is, \( \Lambda^* \) is a Banach space. By Corollary 2 this implies that \( \Lambda \) is a Banach space, contrary to assumption. Thus, \( S(\Lambda^*, \Lambda) \) is not a metric topology.

5. A strong unit \( u \) in a partially ordered vector space \( X \) is an element such that for each \( x \) in \( X \), there exists a constant \( \lambda \) for which \( x \leq \lambda u \).

**Theorem 5.** If the Köthe space \( \Lambda^* \) contains a strong unit, then \( \Lambda \) has a unique Köthe topology, and with this topology \( \Lambda \) is a Banach space.

**Proof.** Let \( u \geq 0 \) be a strong unit in \( \Lambda^* \). Set \( B = \{ g \in \Lambda^* : |g| \leq u \} \). If \( g \) is in \( \Lambda^* \), then a constant \( \lambda \) exists such that \( g \) is in \( \lambda B \), that is, \( B \) is an absorbing subset of \( \Lambda^* \). Clearly the set \( B \) is admissible. Hence \( \Lambda \) is a Banach space for the topology \( S(\Lambda, \Lambda^*) \).

Let \( \kappa_w(\Lambda, \Lambda^*) \) be any Köthe topology for \( \Lambda \). There exists a normal set \( V \) in \( W \) such that \( u \) is in \( V \). Thus \( B \subset V \), from which it follows that \( \kappa_w(\Lambda, \Lambda^*) \) is finer than \( S(\Lambda, \Lambda^*) \). As \( S(\Lambda, \Lambda^*) \) is always finer than \( \kappa_w(\Lambda, \Lambda^*) \), these two topologies must be the same.

The existence of a unique Köthe topology on \( \Lambda \) does not imply
that $\Delta^*$ has a strong unit. For consider the Köthe space $\Omega$ of all real sequences. Its associate, $\Phi$, is the space of sequences which are zero except at a finite number of places. This space has no strong unit. However, $\Omega$ has only one Köthe topology. Note that the weak topology for $\Omega$, that is the topology of pointwise convergence, is also the weakest Köthe topology. This topology is compatible with the order relation in $\Omega$ and is also a metric topology. Hence by [6] or [8] it is the finest compatible topology. Since $S(\Omega, \Phi)$ is a compatible topology it is weaker than the weak topology $\sigma(\Omega, \Phi)$. Thus these two topologies are the same, that is, $\Omega$ has only one Köthe topology.

References


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