CONTINUED FUNCTION EXPANSIONS OF REAL NUMBERS

B. K. SWARTZ AND B. WENDROFF

1. Introduction. We present a theory of continued function expansions of numbers which contains the generalized continued fractions of B. H. Bissinger [1] and the generalized decimal representations of C. J. Everett [2]. The latter used the following algorithm for representing numbers as sequences of integers: for any \( \gamma \geq 0 \) let \( \gamma_0 = \gamma \), \( \gamma_{n+1} = f^{-1}(\gamma_n - a_n) \), where \( a_n = [\gamma_n] \) and \( f \) is strictly increasing and continuous from \([0, \rho]\) onto \([0, 1]\), \( \rho \) an integer. We generalize this, in particular, by admitting a wider class of functions than those of the form \( f^{-1}(x-n) \). O. W. Rechard [3] gave a necessary and sufficient condition that the correspondence between numbers and sequences resulting from Everett’s algorithm be 1-1. This condition appears in our theory as a simple functional relation similar to one considered by Schreier and Ulam [4].

2. The algorithm. The correspondence between numbers and sequences which we are going to describe depends on a collection of intervals and on functions defined on those intervals. More precisely

**Definition.** An algorithm frame, \( A \), consists of the following: an interval \( R \); a subset \( P \) of the integers containing at least two integers; a partition of \( R \) into disjoint intervals \( I_n, n \in P \); a subset \( P_0 \) of \( P \) containing at least two integers such that \( I = \bigcup_{n \in P_0} I_n \) is an interval; intervals \( M_n, n \in P \), homeomorphic to each other such that \( M_n \subseteq I_n \) and \( I_n - M_n \) consists of at most one point; and an interval \( M \) homeomorphic to each \( M_n \) such that \( \bigcup_{n \in P_0} M_n \subseteq M \subset I \).

It follows from the above definition that if \( \{ M_n, n \in P \} \) is part of an algorithm frame then either all the \( M_n \) are open intervals or all are closed on one end, not necessarily the same, because not all the \( I_n \) can be closed and the \( M_n \) are homeomorphic to each other. Also, if any interval is infinite at some end it is taken to be open at that end.

**Definition.** An algorithm basis consists of an algorithm frame \( A \) and a collection of homeomorphisms \( h_n, n \in P \), mapping \( M_n \) onto \( M \). We usually identify an algorithm basis by the couple \( (A, h_n) \).

Corresponding to any algorithm basis we have the following algorithm for relating points in \( R \) to sequences (finite or infinite) of integers:

---

Presented to the Society, January 29, 1960; received by the editors October 9, 1959.

1 Work performed under A.E.C. Contract No. W-7405-Eng. 36.
Let \( x_0 \in R \).

\[
\{a(0) \text{ is determined by the requirement that } x_0 \in I_{a(0)} \}. \\
\]

\( a(0) : \)

\[
\left\{ \begin{array}{l}
\text{If } x_0 \in M_{a(0)}, \text{ stop, and represent } x_0 \text{ by the sequence of one element } \{a(0)\}. \\
\text{Since } x_0 \in M_{a(0)} \text{ we can let } x_1 = h_{a(0)}(x_0). \text{ Then } a(1) \text{ is determined by the requirement that } x_1 \in I_{a(1)}. \\
\text{Furthermore, } a(1) \in P_0 \text{ since } x_1 \in M \subseteq I.
\end{array} \right.
\]

\( a(1) : \)

\[
\left\{ \begin{array}{l}
\text{If } x_1 \in M_{a(1)}, \text{ stop, and represent } x_0 \text{ by the sequence of two elements } \{a(0), a(1)\}. \\
\text{Since } x_k-1 \in M_{a(k-1)} \text{ we can let } x_k = h_{a(k-1)}(x_{k-1}). \text{ Then } a(k) \text{ is determined by the requirement that } x_k \in I_{a(k)}, \text{ and } a(k) \in P_0.
\end{array} \right.
\]

\( a(k) : \)

\[
\left\{ \begin{array}{l}
\text{If } x_k \in M_{a(k)}, \text{ stop, and represent } x_0 \text{ by } \{a(0), \ldots, a(k)\}. \\
\end{array} \right.
\]

This algorithm contains the expansions considered by Bissinger and Everett. Let \( A B \) be the following algorithm basis:

\[
R = [0, \infty), \quad P = \{0, 1, 2, \ldots\}, \quad P_0 = \{1, 2, \ldots\}, \quad I_n = [n, n + 1), \\
M_n = (n, n + 1), \quad I = [1, \infty), \quad M = (1, \infty),
\]

and let \( h_n(x) = f^{-1}(x - n) \) for \( x \in (n, n + 1) \) where \( f \) is a continuous strictly decreasing function mapping \([1, \infty)\) onto \((0, 1)\). This contains Bissinger's expansions. Everett's expansions come from the algorithm basis \( A E \) given by:

\[
R = I = M = [0, p), \quad I_n = M_n = [n, n + 1), \\
\]

\[
P = P_0 = \{0, 1, \ldots, p - 1\},
\]

and \( h_n(x) = f^{-1}(x - n), x \in [n, n + 1) \) where \( f \) is continuous and strictly increasing from \([0, p]\) onto \([0, 1]\).

3. **1-1 Correspondence.** Given an algorithm basis \((A, h_n)\), the algorithm defines a function \( h \) from \( R \) into the space \( C \) of finite or infinite sequences of integers \( c = \{c(0), c(1), \ldots\} \) as follows: let \( x \)
yield \( c \) under the algorithm, then \( h(x) = c \). Let \( E \) be the set of all such functions. In general we will use the convention that if \( g \in E \) then the homeomorphisms in its algorithm basis are \( g_n \).

**Definition.** Let \((A, h_n), (B, g_n)\) be algorithm bases. The corresponding functions \( h \) and \( g \in E \) are said to be equivalent, written \( h \sim g \), if \( A \) and \( B \) are identical and if \( h_n \) has the same sense as \( g_n \) for each \( n \). (By this we mean that if \( h_n \) is monotone increasing so is \( g_n \) and if \( h_n \) is monotone decreasing so is \( g_n \). This is not meant to imply that the sense of \( h_n \) is independent of \( n \).)

Denote by \( C(h) \) the range of \( h \) for \( h \in E \).

The following theorems characterize the equivalent 1-1 functions in \( E \):

**Theorem 1.** If \( h \sim g \) and \( h \) is 1-1 onto \( C(h) \) then \( C(h) \subset C(g) \).

**Corollary 1.** If \( h \sim g \), a finite sequence is in \( C(h) \) if and only if it is in \( C(g) \).

**Corollary 2.** If \( g \) is 1-1, \( C(h) = C(g) \).

**Notation.** A sequence of functions \( h g \cdots k \) always means the composite function \( h(g(\cdots(k)\cdots)) \).

**Theorem 2.** Let \( g \) be 1-1 from \( R \) onto \( C(g) \) and let \( h \) have the same algorithm frame as \( g \). Then \( h \sim g \) and \( h \) is 1-1 from \( R \) onto \( C(g) \) if and only if there exists an increasing homeomorphism \( F \) from \( R \) onto \( R \), which also maps \( M_n \) onto \( M_n \) for all \( n \), such that \( h_n^{-1} = F^{-1} g_n^{-1} F \).

The following theorems are an application of Theorem 2 to bases \( AB \) and \( AE \), respectively.

**Theorem 3.** Let \((A, h_n)\) be an algorithm basis of the form \( AB \). Let \( h_n(x) = \bar{h}^{-1}(x-n) \). Then \( h \) is 1-1 if and only if there exists an increasing homeomorphism \( F \) mapping \([0, \infty)\) onto itself such that \( F(x) = n + F(x-n) \) for \( x \in [n, n+1) \) and \( \bar{h}^{-1}(\tau) = F^{-1}(1/F(\tau)) \) for all \( \tau \in (0, 1] \).

**Theorem 4.** Let \((A, h_n)\) be an algorithm basis of the form \( AE \). Let \( h_n(x) = \bar{h}^{-1}(x-n) \). Then \( h \) is 1-1 if and only if there exists an increasing homeomorphism \( F \) mapping \([0, p]\) onto itself such that \( F(x) = n + F(x-n) \) for \( x \in [n, n+1) \) and \( \bar{h}^{-1}(\tau) = F^{-1}(p \cdot F(\tau)) \) for all \( \tau \in [0, 1] \).

Reichard's condition is that \( h \) is 1-1 if and only if there exists an increasing homeomorphism \( G \) mapping \([0, 1]\) onto itself such that \( \bar{h}(y) = G^{-1}((n+G(y-n))/p) \). It is easily verified that this is equivalent to Theorem 4 (given \( G \), set \( F(y) = n + G(y-n) \), \( y \in [n, n+1) \), and given \( F \) set \( G(\tau) = F(\tau) \), \( \tau \in [0, 1] \)).

**Proof of Theorem 1.**
Lemma. Let \((A, f_n)\) be any algorithm basis and let \(c\) be any infinite sequence \(\{c(0), c(1), \ldots\}\) such that \(c(0) \in P, c(i) \in P_0\) for \(i > 0\). Let \(F_k = f_{c(0)}^{-1} \cdots f_{c(k-1)}^{-1}(M_c) = f_{c(0)}^{-1} \cdots f_{c(k-1)}^{-1}(M_{c(k)})\). Then \(f(x) = c\) if and only if \(x \in \cap_0 G_k\).

Proof of Lemma. \(F_k\) consists exactly of those points \(y\) which correspond, under \(f\), to sequences with at least \(k + 2\) entries, the first \(k + 1\) of which are \(c(0), \ldots, c(k)\), and the lemma follows immediately from this fact. Proceeding with the theorem, let \(h\) be \(1-1\) onto \(C(h)\), \(h \sim g\), and let \(h(x) = c\). If \(c = \{c(0)\}\), then \(g(x) = c\). If \(c = \{c(0), \ldots, c(k)\}\), \(k > 0\), then \(x = h_{c(0)}^{-1} \cdots h_{c(k-1)}^{-1}(y)\) where \(y \in I_{c(k)} - M_{c(k)}\) (note that in the definition of algorithm frame it was assumed that \(I_n - M_n\) consists of at most one point; the reason for this is apparent, for if there were more than one point \(k\) could not be \(1-1\)). Then if \(w = g_{c(0)}^{-1} \cdots g_{c(k)}^{-1}(y)\), \(g(w) = c\). If \(c\) is infinite, \(c = \{c(0), c(1), \ldots\}\), let
\[
H_k = h_{c(0)}^{-1} \cdots h_{c(k)}^{-1}(M) = h_{c(0)}^{-1} \cdots h_{c(k-1)}^{-1}(M_{c(k)}), \quad G_k = g_{c(0)}^{-1} \cdots g_{c(k)}^{-1}(M) = g_{c(0)}^{-1} \cdots g_{c(k-1)}^{-1}(M_{c(k)}),
\]
and
\[
r_k = g_{c(0)}^{-1} \cdots g_{c(k)}^{-1}h_{c(k)} \cdots h_{c(0)}.
\]
Clearly, \(H_{k+1} \subset H_k\), \(G_{k+1} \subset G_k\), \(G_k = r_k(H_k)\), and by the lemma, \(x = \cap_0^\infty H_k\). Furthermore, since \(h \sim g\), there are at most an even number of decreasing homeomorphisms in the composition of \(r_k\), therefore each \(r_k\) is strictly increasing from the interval \(H_k\) onto the interval \(G_k\). Also,
\[
r_k(H_{k+1}) = g_{c(0)}^{-1} \cdots g_{c(k)}^{-1}h_{c(k)} \cdots h_{c(0)}[h_{c(0)}^{-1} \cdots h_{c(k)}^{-1}(M_{c(k+1)})] = G_{k+1}.
\]
It follows from these facts that \(\cap_0^\infty G_k\) is nonempty. To show this we consider three cases.

Case 1. Each \(H_k\) is open. Then each \(G_k\) is open. Let \(H_k = (a_k, b_k)\), \(G_k = (\alpha_k, \beta_k)\). Since \(\cap_0^\infty H_k\) consists of the point \(x\), we must have that \(\lim a_k > a_k\) for all \(k\) and \(\lim b_k < b_k\) for all \(k\) (this also means that if \(b_0 = \infty\) some \(a_k\) must be finite, and similarly, if \(a_0 = -\infty\), some \(a_k\) is finite). Then there must be infinitely many indices \(k\) for which \(a_k < a_{k+1}\). Let \(a_k < a < a_{k+1}\). Then \(\alpha_k < r_k(a) < r_k(a_{k+1}) = \alpha_{k+1}\), and therefore if \(\alpha = \lim \alpha_k, \alpha > a_k\) for all \(k\). By the same kind of reasoning if \(\beta > \beta_k, \beta < \beta_k\) for all \(k\). Since \(\alpha \leq \beta\), \(\cap_0^\infty G_k = [\alpha, \beta]\), nonempty.

Case 2. Each \(H_k\) is closed on one end and \(k_0\) exists such that \(H_k\) is closed on the same end as \(H_{k_0}\), say the left for \(k \geq k_0\). The \(G_k\) must have the same property. Let \(H_k = [a_k, b_k], G_k = [\alpha_k, \beta_k], k \geq k_0\). By the
same reasoning as in Case 1 if \( \beta = \lim \beta_k, \beta < \beta_k \) for all \( k \), therefore \( \bigcap_{0}^{\infty} G_k = \bigcap_{0}^{\infty} [\alpha_k, \beta] \) which is nonempty.

**Case 3.** Each \( H_k \) is closed on one end but no \( k_0 \) as in Case 2 exists. Then it is easily seen that \( \bigcap_{0}^{\infty} G_k = \bigcap_{0}^{\infty} G_k \) which is nonempty. Since in all cases \( \bigcap_{0}^{\infty} G_k \) is nonempty, there exists \( y \in R \) such that \( g(y) = c \), which completes the proof.

The proof of Corollary 1 is essentially contained in the analysis of finite sequences given above. Corollary 2 is immediate.

**Proof of Theorem 2.** Let \( h \sim g \) and both be 1-1 onto \( C(h) = C(g) \). Let \( x \in R \). The following function \( F \) is 1-1 from \( R \) onto \( R \); if \( h(x) = c \) then \( y = F(x) \) if \( g(y) = c \). Since each interval \( M_n \) consists exactly of those points which correspond under the algorithm to sequences containing at least two entries, the first of which is \( n \), \( F \) maps \( M_n \) onto \( M_n \). If \( h(x) = \{c(0)\} \), then \( F(x) = x \) so \( F \) maps \( I_n \) onto \( I_n \). To see that \( F \) is strictly increasing, let \( x < \hat{x}, h(x) = c, h(\hat{x}) = d \). Define the length \( l \) of \( c \) as follows: if \( c = \{c(0), \ldots, c(k)\} \), then \( l = k \), and if \( c \) is infinite \( l = \infty \). Let \( l \) be the length of \( d \). There are two cases to consider.

**Case 1.** There exists an integer \( k \leq \min(l, l) \) such that \( c(i) = d(i), i < k, \) and \( c(k) \neq d(k) \). If \( k = 0 \), since \( x \in I_{c(0)}, \hat{x} \in I_{d(0)} \), we must have that \( I_{c(0)} \) is to the left of \( I_{d(0)} \). Since \( F(x) \in I_{c(0)} \), \( F(\hat{x}) \in I_{d(0)} \), \( F(x) < F(\hat{x}) \). If \( k > 0 \) then we can write

\[
x = h_{c(0)}^{-1} \cdots h_{c(k-1)}^{-1}(x_k)
\]
for some \( x_k \in I_{c(k)} \),

\[
\hat{x} = h_{c(0)}^{-1} \cdots h_{c(k-1)}^{-1}(\hat{x}_k)
\]
for some \( \hat{x}_k \in I_{d(k)} \),

\[
F(x) = g_{c(0)}^{-1} \cdots g_{c(k-1)}^{-1}(y_k)
\]
for some \( y_k \in I_{c(k)} \)

\[
F(\hat{x}) = g_{c(0)}^{-1} \cdots g_{c(k-1)}^{-1}(\hat{y}_k)
\]
for some \( \hat{y}_k \in I_{d(k)} \).

Let \( h_{c(0)}^{-1} \cdots h_{c(k-1)}^{-1} \) be increasing. Then \( x_k < \hat{x}_k, I_{c(k)} \) is to the left of \( I_{d(k)} \), \( y_k < \hat{y}_k \) and therefore \( F(x) < F(\hat{x}) \) since \( g_{c(0)}^{-1} \cdots g_{c(k-1)}^{-1} \) is also increasing. If \( h_{c(0)}^{-1} \cdots h_{c(k-1)}^{-1} \) is decreasing then \( x_k > \hat{x}_k, I_{c(k)} \) is to the right of \( I_{d(k)} \), \( y_k > \hat{y}_k \) and \( F(x) < F(\hat{x}) \) since \( g_{c(0)}^{-1} \cdots g_{c(k-1)}^{-1} \) is also decreasing.

**Case 2.** \( \min(l, l) \) is finite and \( c(i) = d(i), i \leq \min(l, l) \). For definiteness let \( l < l \). If \( l = 0 \) then \( x \in I_{c(0)} \) - \( M_{c(0)} \). Since \( x \in M_{c(0)}, x \) is the left end point of \( I_{c(0)} \). Then \( F(x) = x < F(\hat{x}) \) because \( F(\hat{x}) \in M_{c(0)} \). If \( l > 0 \) we can write

\[
x = h_{c(0)}^{-1} \cdots h_{c(l-1)}^{-1}(x_l), \quad x_l \in I_{c(l)} - M_{c(l)},
\]

\[
\hat{x} = h_{c(0)}^{-1} \cdots h_{c(l-1)}^{-1}(\hat{x}_l), \quad \hat{x}_l \in M_{c(l)},
\]

\[
F(x) = g_{c(0)}^{-1} \cdots g_{c(l-1)}^{-1}(x_l),
\]

\[
F(\hat{x}) = g_{c(0)}^{-1} \cdots g_{c(l-1)}^{-1}(\hat{y}_l), \quad \hat{y}_l \in M_{c(l)}.
\]
If $h_{(0)}^{-1} \cdots h_{(i-1)}^{-1}$ is increasing, $x_i < x_i$, therefore $x_i$ is the left end-point of $I_{(i)}$, therefore $x_i < y_i$ and $F(x) < F(x)$. The proof is straightforward if $h_{(0)}^{-1} \cdots h_{(i-1)}^{-1}$ is decreasing. Thus $F$ is strictly increasing and is therefore a homeomorphism of $R$ onto $R$. Let $x \in M_n$. Then $x = h_n^{-1}(r)$ and $F(x) = g_n^{-1}(s)$. But $s = F(r)$ (this follows from the fact that if $h(x) = \{a_1, a_2, \cdots \}$ and $x = h_{(n)}(r)$ then $h(r) = \{a_1, a_2, \cdots \}$) therefore $F(x) = g_n^{-1}(F(h_n(x)))$ or $h_n^{-1} = F^{-1}g_n^{-1}F$. Conversely let $h_n^{-1} = F^{-1}g_n^{-1}F$ where $F$ is an increasing homeomorphism from $R$ onto $R$ taking $M_n$ onto $M_n$. Let $y = F(x)$. Then $h(x) = c$ if and only if $g(y) = c$, which completes the proof.

Proof of Theorem 3 and Theorem 4. Theorem 3 is obtained simply by applying Theorem 2 to this basis, using the inverse functional relation $h_n = F^{-1}g_n F$ and choosing $g_n(x) = 1/(x-n)$ (the corresponding $g$ is the ordinary continued fraction algorithm which is well-known to be 1-1). Theorem 4 is obtained by taking $g_n(x) = p \cdot (x-n)$ (the corresponding $g$ is the ordinary decimal expansion to the base $p$, which is 1-1). In both cases the functional relation implies that $F(x) - n$ is a function of $x - n$ only and therefore $F(x) - n = F(x - n)$ for $x \in [n, n+1)$.

Finally, let $(A, h_n)$ be an algorithm basis giving rise to the function $h \in E$ and suppose $h$ is 1-1. If $x \in R$ and $c$ is an infinite sequence such that $h(x) = c$, there are two ways of interpreting the continued function expansion of $x$:

$$x = h_{(0)}^{-1}(h_{(1)}^{-1}(\cdots)).$$

The first is that for every $k \geq 0$, $x = h_{(0)}^{-1} \cdots h_{(k)}^{-1}(y)$ where $h(y) = \{c(k+1), \cdots \}$. The second is that $x = \lim_{k \to \infty} h_{(0)}^{-1} \cdots h_{(k)}^{-1}(y)$ for all $y \in M$, which follows from the fact that $x = \bigcap_0^\infty H_k$.

Bibliography


Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico